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Graphs with maximum size and given paired-domination number

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ABSTRACT

A paired-dominating set of a graph is a dominating set whose induced subgraph contains a perfect matching. The paired-domination number of a graph G is the minimum cardinality of a paired-dominating set in G. We determine the maximum possible number of edges in a graph with given order and given paired-domination number and we completely characterize the infinite family of graphs that achieve this maximum possible size. Our result builds on a classic result in 1962 due to Erdős and Rényi (1962) since the case when the paired-domination is four is equivalent to determining the minimum size of a nontrivial diameter-2 graph (excluding a star) in the complement of the graphs we are considering. More precisely, for $k \geq 2$, let G be a graph with paired-domination number $2k$, order $n \geq 2k$, and size m. As a consequence of the Erdős–Rényi result it follows that if $k = 2$, then $m \leq \binom{n}{2} - k(n-2) + 1$. For $k \geq 3$, we show that $m \leq \binom{n}{2} - k(n-2)$ and we characterize the graphs that achieve equality in this bound.

1. Introduction

In this paper we continue the study of paired-domination in graphs. Domination and its variations in graphs are now well-studied and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [16,17]. Paired-domination was introduced by Haynes and Slater [19,20] as a model for assigning backups to guards for security purposes. Recent papers in this topic can be found, for example, in [2,3,6,7,4,5,9,10,12–15,18,22,24,23,26], and elsewhere.

Let G be a graph. A *matching* in G is a set of independent edges in G, while a *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M. A *dominating set* of G is a set S of vertices of G such that every vertex not in S is adjacent to some vertex in S. A *paired-dominating set*, abbreviated PD-set, of a graph G is a dominating set S of G with the additional property that the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). Two vertices joined by an edge of M are said to be *paired* and are also called *partners* in S. Every graph without isolated vertices has a PD-set since the end-vertices of any maximal matching form such a set. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G, while the *paired-domination number* of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a PD-set in G. A PD-set of cardinality $\gamma_{pr}(G)$ we call a $\gamma_{pr}(G)$-set.

A classical result of Vizing [30] relates the size and the ordinary domination number, γ, of a graph of given order. Rautenbach [27] shows that the square dependence on n and γ in the result of Vizing turns into a linear dependence on n, γ, and the maximum degree Δ. Dankelmann et al. [8] proved a Vizing-like relation between the size and the total domination number of a graph of given order. Sanchis [28] showed that if we restrict our attention to connected graphs with total

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domination number at least 5, then the bound in [8] can be improved slightly. The square dependence on n and γ_t presented in [8,28] is improved in [21,29,31] into a linear dependence on n, γ_t and Δ by demanding a more even distribution of the edges by restricting the maximum degree Δ.

In this paper we present a Vizing-like relation between the size and the paired-domination number of a connected graph. More precisely, we determine the maximum possible number of edges in a graph with given order and given paired-domination number. Further, we completely characterize the infinite family of graphs that achieve this maximum possible size. Our proof method uses intricate counting arguments and relies heavily on result in both matching theory and domination theory as well as surprisingly on a classic result in 1962 due to Erdős and Rényi [11] on the minimum size of a nontrivial diameter-2 graph (excluding a star). Our result builds on the Erdős–Rényi result since the case when the paired-domination is four is equivalent to determining the minimum size of a nontrivial diameter-2 graph in the complement of the graphs we are considering.

1.1. Notation

For notation and graph theory terminology not defined herein, we refer the reader to [16]. Let $G = (V, E)$ be a graph with vertex set V of order $n(G) = |V|$ and edge set E of size $m(G) = |E|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v\in S} N(v)$, and its *closed neighborhood* is the set $N_G[S] = N(S) \cup S$. If the graph G is clear from the context, we simply write $d(v)$, $N(v)$, $N[v]$, $N(S)$ and $N[S]$ rather than $d_G(v)$, $N_G(v)$, $N_G[v]$, $N_G(S)$ and $N_G[S]$, respectively. We denote the complement of G by $\overline{G}$.

A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A *strong support vertex* is adjacent to at least two leaves. A neighbor of a vertex v that is a leaf we call a *leaf-neighbor* of v. A *star* is the tree $K_{1,n-1}$ of order $n \geq 2$. A *subdivided star* is a star where each edge is subdivided exactly once. A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. A component of a graph G isomorphic to a graph F we call an *F-component* of G. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. Further if $S \neq V$, then we denote the graph obtained from G by deleting all vertices in S (as well as all incident edges) by $G - S$. Let $v \in V$. If $S = \{v\}$, we simply denote $G - S$ by $G - v$. We let $N_S(v) = N(v) \cap S$ and $d_S(v) = |N_S(v)|$, and so $d_S(v)$ denotes the number of vertices in S adjacent to v in G. In particular, $d_V(v) = d_G(v)$.

Let X and Y be subsets of vertices of G. We denote the set of edges that join a vertex of X and a vertex of Y in G by $G[X, Y]$, or simply by $[X, Y]$ if the graph G is clear from context. Thus, $|[X, Y]|$ is the number of edges with one end in X and the other end in Y. In particular, $|[X, X]| = m(G[X])$. If all possible edges in $[X, Y]$ are present, we say that $[X, Y]$ is *full*. We say that the set X *dominates* the set Y if $Y \subseteq N[X]$. A vertex of G adjacent to every other vertex in G we call a *dominating vertex* of G.

For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ between u and v is the length of a shortest u–v path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, which is denoted by diam(G). We say that G is a *diameter-2 graph* if diam(G) = 2.

2. Special families of graphs

In this section we define several families of graphs. Let $C_5(n_1, n_2, n_3, n_4, n_5)$ denote the graph that can be obtained from a 5-cycle $x_1x_2x_3x_4x_5$ by replacing each vertex x_i, $1 \leq i \leq 5$, with a nonempty clique X_i, where $|X_i| = n_i \geq 1$, and adding all edges between X_i and X_{i+1}, where addition is taken modulo 5. For $n \geq 5$, let $\mathcal{F}_n = \{C_5(n_1, n_2, n_3, n_4, n_5) \mid n_1 = n_2 = n_3 = 1 \text{ and } n = n_4 + n_5 + 3\}$. A graph in the family $\mathcal{F}_n$ is illustrated in Fig. 1, where in this diagram both A and B represent cliques, $[A, B]$ is full, the vertex x_1 dominates $A \cup \{x_2\}$, the vertex x_3 dominates $B \cup \{x_2\}$ and where $n = |A| + |B| + 3$.

For $n \geq 7$, let $\mathcal{G}_n$ be the family of graphs constructed as follows: Take a complete graph on $n - 4$ vertices with vertex set S and partition the set S into three (nonempty) sets, A, B and C. Add three new vertices, a, b and c, and join a to every vertex in $S \setminus A$, join b to every vertex in $S \setminus B$, and join c to every vertex in $S \setminus C$. Finally add a new vertex, v, and join v to a, b and c. A graph in the family $\mathcal{G}_n$ is illustrated in Fig. 1. In this diagram A, B and C represent cliques and $[A, B]$, $[A, C]$ and $[B, C]$ are full. The vertex a dominates $B \cup C \cup \{v\}$, the vertex b dominates $A \cup C \cup \{v\}$ and the vertex c dominates $A \cup B \cup \{v\}$.

If $k \geq 3$ and $n = 2k$, let $R_{n,k}$ be the graph, kP_2, consisting of k disjoint copies of P_2. For $k \geq 3$ and $n \geq 2k + 1$, let $R_{n,k}$ be the graph obtained from the disjoint union of k copies of P_2 and a copy of the complete graph on $n - 2k$ vertices by joining one vertex from each copy of P_2 to every vertex in the complete graph. For $k \geq 3$, we note that $\gamma_{\rm pr}(R_{n,k}) = 2k$ and that $R_{n,k}$ has size $\binom{n}{2} - k(n - 2)$. The graph $R_{n,k}$ is illustrated in Fig. 2.

3. Main results

Our aim in this paper is twofold. Firstly, to determine the maximum possible number of edges in a graph with given order and given paired-domination number. Secondly, to completely characterize the infinite family of graphs that achieve this maximum possible size. We begin with the following trivial observation.

Observation 1. *Let G be a graph of order $n \geq 2$ and size m satisfying $\gamma_{\rm pr}(G) = 2$. Then, $m \leq \binom{n}{2}$, with equality if and only if $G = K_n$.*