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Bounding the sum of the largest Laplacian eigenvalues of graphs

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ABSTRACT

We prove that Brouwer's conjecture holds for certain classes of graphs. We also give upper bounds for the sum of the largest Laplacian eigenvalues for graphs satisfying certain properties: those that contain a path or a cycle of a given size, graphs with a given matching number and graphs with a given maximum degree. Then we provide conditions for which these upper bounds are better than the previous known results.

1. Introduction

In this paper, we investigate the sum of the largest Laplacian eigenvalues of a graph, which is the subject of a conjecture proposed by A.E. Brouwer and which has recently, in [3,4], showed to be helpful for the study of the Laplacian energy of graphs.

Given graph G on the vertex set $V = \{v_1, \dots, v_n\}$, the *Laplacian matrix* of G is given by $L = D - A$, where D is the diagonal matrix whose entry (i, i) is equal to the degree of v_i and A is the adjacency matrix of G. The *Laplacian spectrum* of G is defined as the set of eigenvalues of L, which we shall denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. With this, the sum of the largest Laplacian eigenvalues of G is denoted by

$$S_k(G) = \sum_{i=1}^{n} \mu_i.$$

There are many results concerning $S_k(G)$. In [8], there are upper bounds for $S_k(G)$ in terms of the number of vertices and edges. Also, the author characterizes the extremal cases.

Brouwer proposed the following conjecture.

Conjecture 1. *Let $G = (V, E)$ with n vertices. Then*

$$S_k(G) \leq |E| + \binom{k+1}{2}, \tag{1}$$

for $1 \leq k \leq n$.

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In [1], some advances on Conjecture 1 are mentioned: this inequality holds for split graphs, in particular for threshold graphs. It also holds for regular graphs. For $k = 1$, Conjecture 1 follows from the well-known inequality $\mu_1(G) \leq n$. In [5], Haemers, Mohammadian and Tayfeh-Rezaie showed that Conjecture 1 is true for all graphs when $k = 2$ and is also true for trees. We say that a connected graph is c-cyclic, or it has c cycles, if it has $n - 1 + c$ edges and we say that the graph is unicyclic or bicyclic if it is 1-cyclic or 2-cyclic respectively. More recently, Du and Zhou [2] showed that the conjecture holds for unicyclic and bicyclic graphs.

The authors of [2] have also proved the following lemma.

Lemma 1. *Let H be a subgraph of graph G, and $|V(H)| = n_1 \geq 2$. Then*

$$S_k(G) \leq S_k(H) + 2(e(G) - e(H))$$

for $1 \leq k \leq n_1$.

This lemma provides a useful tool to extend an upper bound on trees for the general class of connected graphs. Assume that we have a new upper bound for a tree, that is, $S_k(T)$ is bounded for any tree T. Now consider a connected graph G with n vertices. Then for any generating tree T of G using Lemma 1, we obtain the following result.

Lemma 2. *Let G be a connected graph on n vertices and let T be a generating tree of G. Then*

$$S_k(G) \leq S_k(T) + 2(e(G) - n + 1)$$

for $1 \leq k \leq n$.

Therefore, for the purpose of bounding S_k for graphs, it may be interesting to start investigating this parameter for trees. In [3], the following upper bound is given for a tree T on n vertices.

$$S_k(T) \leq n - 2 + 2k - \frac{2k-2}{n}. \tag{2}$$

By applying this upper bound for trees, the authors of [6] proved the following upper bound for a connected graph G on n vertices.

$$S_k(G) \leq 2e(G) - n + 2k - \frac{2k-2}{n}. \tag{3}$$

They also settled Brouwer's conjecture for unicyclic and bicyclic graphs, since for such graphs the upper bound (3) is tighter than (1).

However, for c-cyclic graphs, whenever $c \geq 3$ the upper bound (3) is not enough to settle Brouwer's conjecture. For the general case, in [2] the following result was proved.

Lemma 3. *Let G be a graph with n vertices, of which n_1 are not isolated vertices. Then inequality (1) holds for $1 \leq k \leq n$ if $9 - 8(n_1 - e(G)) < 0$, and for $\left\lceil \frac{3+\sqrt{9-8(n_1-e(G))}}{2} \right\rceil \leq k \leq n$ if $9 - 8(n_1 - e(G)) \geq 0$.*

Lemma 3 will play a special role in our results. Also, in [6] the following result was proved.

Lemma 4. *Let G be a connected graph on n vertices. Then inequality (1) holds for*

$$\frac{3n - 4 + \sqrt{9n^2 - 8n + 16 + 8e(G)n^2 - 8n^3}}{2n} \leq k \leq n.$$

The bound (2) is tight when $k = 1$ and T is a star, and it cannot be improved by subtracting $1/n$, even if we consider trees with diameter at least three [3]. However, in [4], the authors have shown that the upper bound (2) could be improved by $2/n$ for trees with diameter at least 4 and at least 6 vertices.

$$S_k(T) \leq n - 2 + 2k - \frac{2k}{n}. \tag{4}$$

This slight improvement bounding (2) was sufficient to obtain a sharper result than Lemma 4 and, consequently, new families of graphs satisfying (1), as showed in [4].

Naturally, Brouwer's conjecture holds in case it is possible to show an upper bound tighter than (1). In an attempt to achieve such sharper bounds, we shall estimate $S_k(G)$ for graphs depending on certain parameters, such as diameter, maximum degree, matching number and girth. These results allow us to settle Brouwer's conjecture for such graphs or at least to prove that inequality (1) holds for some values of k. By obtaining a sharp bound for paths, we also give upper bounds for the sum of the largest Laplacian for graphs containing a path or a cycle of a given size.

This paper is organized as follows. In Section 2 we study and bound $S_k(G)$ for G satisfying certain conditions and in Section 3 we show new families of graphs for which Brouwer's conjecture holds. We finish by comparing the bounds with existing bounds.