Discrete Applied Mathematics 161 (2013) 1118–1127

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

An extension of A.R. Rao's characterization of potentially K_{m+1}-graphic sequences☆

Jian-Hua Yin *

Department of Mathematics, College of Information Science and Technology, Hainan University, Haikou 570228, PR China

ARTICLE INFO

Article history:
Received 21 October 2011
Received in revised form 3 December 2012
Accepted 7 December 2012
Available online 25 December 2012

Keywords:
Complete graph
Degree sequence
A.R. Rao's characterization

ABSTRACT

Let $A_n = (a_1, a_2, \dots, a_n)$ and $B_n = (b_1, b_2, \dots, b_n)$ be two sequences of nonnegative integers with $a_1 \ge a_2 \ge \cdots \ge a_n$, $a_i \le b_i$ for $i = 1, 2, \dots, n$ and $a_i + b_i \ge a_{i+1} + b_{i+1}$ for $i = 1, 2, \dots, n-1$. $(A_n; B_n)$ is said to be *potentially* K_{m+1} (resp. A_{m+1})*-graphic* if there exists a simple graph G with vertices $v_1, v_2, \dots, v_n$ such that $a_i \le d_G(v_i) \le b_i$ for $i = 1, 2, \dots, n$ and G contains K_{m+1} as a subgraph (resp. the induced subgraph of $\{v_1, v_2, \dots, v_{m+1}\}$ in G is K_{m+1}), where K_{m+1} is the complete graph on $m+1$ vertices. In this paper, we give a good characterization of $(A_n; B_n)$ that is potentially A_{m+1}-graphic. As a corollary, we also obtain a good characterization of $(A_n; B_n)$ that is potentially K_{m+1}-graphic if $b_1 \ge b_2 \ge \cdots \ge b_n$. This is an extension of A.R. Rao's characterization of potentially K_{m+1}-graphic sequences.

1. Introduction

A non-increasing sequence $\pi = (d_1, d_2, \dots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The following well-known theorem due to Erdős and Gallai [2] gives a good characterization of π that is graphic.

Theorem 1.1 (*[2]*). *Let $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of nonnegative integers. Then π is graphic if and only if $\sum_{i=1}^{n} d_i$ is even and*

$$\sum_{i=1}^{t} d_i \le t(t-1) + \sum_{i=t+1}^{n} \min\{t, d_i\} \quad \text{for each } t \text{ with } 1 \le t \le n.$$

A non-increasing sequence $\pi = (d_1, d_2, \dots, d_n)$ of nonnegative integers is said to be *potentially K_{m+1}-graphic* if there is a realization of π containing K_{m+1} as a subgraph. A.R. Rao [7] proved that π is potentially K_{m+1}-graphic if and only if π has a realization containing K_{m+1} on those vertices having degrees $d_1, d_2, \dots, d_{m+1}$. In [8], A.R. Rao gave a good characterization of π that is potentially K_{m+1}-graphic. This is a generalization of Theorem 1.1 (which corresponds to $m = 0$).

Theorem 1.2 (*[8]*). *Let $n \ge m+1$ and $\pi = (d_1, d_2, \dots, d_n)$ be a non-increasing sequence of nonnegative integers with $d_{m+1} \ge m$. Then π is potentially K_{m+1}-graphic if and only if $\sum_{i=1}^{n} d_i$ is even and*

$$\sum_{i=1}^{p} (d_i - m) + \sum_{i=m+2}^{q+m+1} d_i \le 2pq + q(q-1) + \sum_{i=p+1}^{m+1} \min\{q, d_i - m\} + \sum_{i=q+m+2}^{n} \min\{p+q, d_i\} \qquad (1)$$

for all p and q with $0 \le p \le m+1$ and $0 \le q \le n-m-1$.

☆ Supported by National Natural Science Foundation of China (Grant Nos. 11161016, 11261015 and 10861006).
* Tel.: +86 0898 66288382; fax: +86 0898 66288382.
E-mail addresses: yinjh@ustc.edu, yinjh@hainu.edu.cn.

0166-218X/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2012.12.008

In [8], A.R. Rao gave a lengthy induction proof of Theorem 1.2 via linear algebraic techniques that remains unpublished, but Kézdy and Lehel [4] have given another proof using network flows or Tutte's f-factor theorem. Recently, Yin [9] obtained a short constructive proof of Theorem 1.2.

Let $A_n = (a_1, a_2, \dots, a_n)$ and $B_n = (b_1, b_2, \dots, b_n)$ be two sequences of nonnegative integers with $a_1 \ge a_2 \ge \cdots \ge a_n$, $a_i \le b_i$ for $i = 1, 2, \dots, n$ and $a_i + b_i \ge a_{i+1} + b_{i+1}$ for $i = 1, 2, \dots, n-1$. $(A_n; B_n)$ is said to be *graphic* if there exists a simple graph G with vertices $v_1, v_2, \dots, v_n$ such that $a_i \le d_G(v_i) \le b_i$ for $i = 1, 2, \dots, n$. In [1], Cai, Deng and Zang presented a good characterization of $(A_n; B_n)$ that is graphic. This solves a research problem posed by Niessen [6] and extends Theorem 1.1 (which corresponds to $a_i = b_i = d_i$ for $i = 1, 2, \dots, n$). They defined for $t = 0, 1, \dots, n$

$$I_t = \{i | i \ge t+1 \text{ and } b_i \ge t+1\}$$

and

$$\varepsilon(t) = \begin{cases} 1 & \text{if } a_i = b_i \quad \text{for all } i \in I_t \quad \text{and} \quad \sum_{i \in I_t} b_i + t|I_t| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.3 (*[1]*). *$(A_n; B_n)$ is graphic if and only if*

$$\sum_{i=1}^{t} a_i \le t(t-1) + \sum_{i=t+1}^{n} \min\{t, b_i\} - \varepsilon(t) \quad \textit{for each } t \textit{ with } 0 \le t \le n.$$

Recently, Garg et al. [3] provided a direct and constructive proof of Theorem 1.3. $(A_n; B_n)$ is said to be *potentially K_{m+1}-graphic* if there exists a simple graph G with vertices $v_1, v_2, \dots, v_n$ such that $a_i \le d_G(v_i) \le b_i$ for $i = 1, 2, \dots, n$ and G contains K_{m+1} as a subgraph. Moreover, $(A_n; B_n)$ is said to be *potentially A_{m+1}-graphic* if there exists a simple graph G with vertices $v_1, v_2, \dots, v_n$ such that $a_i \le d_G(v_i) \le b_i$ for $i = 1, 2, \dots, n$ and the induced subgraph of $\{v_1, v_2, \dots, v_{m+1}\}$ in G is K_{m+1}. The purpose of this paper is to investigate a good characterization of $(A_n; B_n)$ that is potentially K_{m+1}-graphic. We first give a good characterization of $(A_n; B_n)$ that is potentially A_{m+1}-graphic as follows. For $0 \le p \le m+1$ and $0 \le q \le n-m-1$, we define $H(p, q) = \{i | p+1 \le i \le m+1 \text{ and } b_i - m \ge q+1\}$ and $I(p, q) = \{i | i \ge q+m+2 \text{ and } b_i \ge p+q+1\}$. If $I(p, q) \ne \emptyset$, we define

$$\varepsilon(p, q) = \begin{cases} 1 & \text{if } a_i = b_i \quad \text{for all } i \in H(p,q) \cup I(p,q) \quad \text{and} \quad \sum_{i \in H(p,q)} (b_i - m) + \sum_{i \in I(p,q)} b_i \\ & \quad + q|H(p,q)| + (p+q)|I(p,q)| \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

and if $I(p, q) = \emptyset$, we define

$$\varepsilon(p, q) = |\{i | i \in H(p, q), a_i = b_i \text{ and } (b_i - m) + q \text{ is odd}\}|.$$

Theorem 1.4. *Let $n \ge m+1$ and $a_{m+1} \ge m$. Then $(A_n; B_n)$ is potentially A_{m+1}-graphic if and only if*

$$\sum_{i=1}^{p} (a_i - m) + \sum_{i=m+2}^{q+m+1} a_i \le 2pq + q(q-1) + \sum_{i=p+1}^{m+1} \min\{q, b_i - m\} + \sum_{i=q+m+2}^{n} \min\{p+q, b_i\} - \varepsilon(p, q) \tag{2}$$

for all p and q with $0 \le p \le m+1$ and $0 \le q \le n-m-1$.

If we add the condition $b_1 \ge b_2 \ge \cdots \ge b_n$, then we can show the following Theorem 1.5.

Theorem 1.5. *If $b_1 \ge b_2 \ge \cdots \ge b_n$, then $(A_n; B_n)$ is potentially K_{m+1}-graphic if and only if it is potentially A_{m+1}-graphic.*

Combining Theorem 1.4 with Theorem 1.5, we have the following Corollary 1.1.

Corollary 1.1. *Let $n \ge m+1$, $a_{m+1} \ge m$ and $b_1 \ge b_2 \ge \cdots \ge b_n$. Then $(A_n; B_n)$ is potentially K_{m+1}-graphic if and only if (2) holds for all p and q with $0 \le p \le m+1$ and $0 \le q \le n-m-1$.*

Remark. We can see that Theorem 1.2 is a corollary of Corollary 1.1. Indeed, setting $a_i = b_i = d_i$ for $i = 1, 2, \dots, n$ in (2) yields

$$\sum_{i=1}^{p} (d_i - m) + \sum_{i=m+2}^{q+m+1} d_i \le 2pq + q(q-1) + \sum_{i=p+1}^{m+1} \min\{q, d_i - m\} + \sum_{i=q+m+2}^{n} \min\{p+q, d_i\} - \varepsilon(p, q) \tag{3}$$

for all p and q with $0 \le p \le m+1$ and $0 \le q \le n-m-1$.