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Deterministic algorithms for multi-criteria Max-TSP

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ABSTRACT

We present deterministic approximation algorithms for the multi-criteria maximum traveling salesman problem (Max-TSP). Our algorithms are faster and simpler than the existing randomized algorithms.

We devise algorithms for the symmetric and asymmetric multi-criteria Max-TSP that achieve ratios of $1/2k - \varepsilon$ and $1/(4k - 2) - \varepsilon$, respectively, where k is the number of objective functions. For two objective functions, we obtain ratios of $3/8 - \varepsilon$ and $1/4 - \varepsilon$ for the symmetric and asymmetric TSP, respectively. Our algorithms are self-contained and do not use existing approximation schemes as black boxes.

1. Multi-criteria TSP

An instance of the traveling salesman problem (TSP) is a complete graph $G = (V, E)$ with edge weights $w : E \to \mathbb{Q}_+$. The goal is to find a *Hamiltonian cycle* (also called a *tour*) of minimum or maximum weight, where the weight of a tour is the sum of its edge weights. (The weight of an arbitrary set of edges is defined analogously.) If G is undirected, we have *Min-STSP* and *Max-STSP* (symmetric TSP). If G is directed, we have *Min-ATSP* and *Max-ATSP* (asymmetric TSP). For Min-ATSP and Min-STSP, we assume that the edge weights fulfill the triangle inequality, since otherwise the two problems cannot be approximated at all (assuming P $\neq$ NP). All these variants of TSP are NP-hard and APX-hard [3]. Thus, we are in need of approximation algorithms. Table 1 shows the currently best approximation ratios for the four variants of the TSP.

In many scenarios, however, there is more than one objective function to optimize. In case of the TSP, we might want to minimize travel time, expenses, number of flight changes, etc., while we want to maximize, e.g., our profit along the route. This gives rise to multi-criteria TSP, where Hamiltonian cycles are sought that optimize several objectives simultaneously. In order to transfer the notion of optimal solutions to multi-criteria optimization problems, *Pareto curves* have been introduced (cf. [7]). A Pareto curve is a set of all optimal trade-offs between the different objective functions.

In the following, k always denotes the number of objective functions. We assume throughout the paper that $k \geq 2$ is an arbitrary constant. Let $[k] = \{1, 2, \ldots, k\}$. The k-criteria variants of the TSP that we consider are denoted by k-Min-STSP and k-Min-ATSP as well as k-Max-STSP and k-Max-ATSP.

We define the following terms for Max-TSP only. After that, we briefly point out the differences for Min-TSP. For a k-criteria variant of Max-TSP, we have edge weights $w_1, \ldots, w_k : E \to \mathbb{Q}_+$. For convenience, let $w = (w_1, \ldots, w_k)$. Inequalities of vectors are meant component-wise. A tour H *dominates* another tour $\tilde{H}$ if $w(H) \geq w(\tilde{H})$ and at least one of these k inequalities is strict. This means that H is strictly preferable to $\tilde{H}$. A *Pareto curve* is a set of all solutions that are not dominated by another solution. Since Pareto curves for the TSP cannot be computed efficiently, we have to be satisfied with approximate Pareto curves. A set $\mathcal{P}$ of tours is called an *α-approximate Pareto curve* for the instance (G, w) if the following holds: For every tour $\tilde{H}$ of G, there exists a tour $H \in \mathcal{P}$ of G with $w(H) \geq \alpha w(\tilde{H})$. We have $\alpha \leq 1$, and a 1-approximate

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Table 1
Approximation ratios for single-criterion and multi-criteria TSP.

Variant	Single-criterion		Multi-criteria		
	Randomized	Deterministic	Randomized	Deterministic	New
Min-STSP		3/2 [3]		$2+\varepsilon$ [14] $2\ (k=2)$ [9]	
Min-ATSP	$O\left(\frac{\log n}{\log\log n}\right)$ [2]	$\frac{2}{3}\cdot\log_2 n$ [8]	$\log n+\varepsilon$ [13] $O\left(\frac{k\log n}{\log\log n}\right)$ [2]	$\frac{2}{3}\cdot k\log_2 n$ [8]	
Max-STSP		7/9 [16]	2/3 [10]		$\frac{1}{2k}-\varepsilon$
				$\frac{7}{27}\ (k=2)$ [13]	$\frac{3}{8}-\varepsilon\ (k=2)$
Max-ATSP		2/3 [11]	1/2 [10]		$\frac{1}{4k-2}-\varepsilon$
					$\frac{1}{4}-\varepsilon\ (k=2)$

Pareto curve is a Pareto curve. An algorithm is called an α *approximation algorithm* if it computes an α-approximate Pareto curve. A *fully polynomial time approximation scheme* (FPTAS) for a multi-criteria maximization problem computes $(1-\varepsilon)$-approximate Pareto curves in time polynomial in the size of the instance and $1/\varepsilon$ for all $\varepsilon > 0$.

For Min-TSP, a tour H dominates $\tilde{H}$ if $w(H) \leq w(\tilde{H})$ and at least one inequality is strict. A set $\mathcal{P}$ of tours is an α-approximate Pareto curve if, for every tour $\tilde{H}$, we have an $H \in \mathcal{P}$ with $w(H) \leq \alpha w(\tilde{H})$. Note that $\alpha \geq 1$ for minimization problems. An FPTAS is a $(1+\varepsilon)$ approximation algorithm.

1.1. Previous work

Table 1 shows the current approximation ratios for the different variants of multi-criteria TSP. Many of these approximation algorithms can be extended to the case where some objectives should be minimized and others should be maximized [12]. We remark that an α approximation for Min-ATSP or Min-STSP yields a $k\alpha$ approximation for k-Min-ATSP or k-Min-STSP simply by encoding all objective functions into a single one. Thus, Feige and Singh's algorithm [8] yields a deterministic $\frac{2}{3}\cdot k\log_2 n$ approximation for k-Min-ATSP and Asadpour et al.'s algorithm [2] yields a randomized $O(k\frac{\log n}{\log\log n})$ approximation.

Unfortunately, no deterministic algorithms are known except for k-Min-STSP, k-Min-ATSP, and 2-Max-STSP. The reason for this is that most approximation algorithms for multi-criteria TSP use cycle covers. A *cycle cover* of a graph is a set of vertex-disjoint cycles such that every vertex is part of exactly one cycle. Hamiltonian cycles are special cases of cycle covers that consist of just one cycle. In contrast to Hamiltonian cycles, cycle covers of optimal weight can be computed in polynomial time. Cycle covers are among the main tools for designing approximation algorithms for the TSP [5,11,8,16,4,6]. However, only a *randomized* fully polynomial-time approximation scheme (FPTAS) for multi-criteria cycle covers is known [18]. This randomized FPTAS builds on a reduction to a specific unweighted matching problem [17], which is then solved using the RNC algorithm by Mulmuley et al. [15]. Derandomizing this algorithm seems to be difficult [1], and these nested reductions make the algorithm quite slow. Hence, it is natural to ask whether there exist deterministic, faster approximation algorithms for multi-criteria TSP.

1.2. New results

We present deterministic approximation algorithms for multi-criteria Max-TSP, which are self-contained and considerably simpler and faster than the existing randomized algorithms. (Table 1 shows an overview.) Our algorithms do not use other algorithms as black boxes except for maximum-weight matching with a single objective function. Furthermore, they do not make any assumption about the representation of the edge weights. The existing algorithms require the (admittedly weak and natural) assumption that the edge weights are encoded in binary.

For k-Max-ATSP, we get a ratio of $\frac{1}{4k-2}-\varepsilon$ for any $\varepsilon > 0$ (Section 2). For k-Max-STSP, we achieve a ratio of $\frac{1}{2k}-\varepsilon$ (Section 3). For the special case of two objective functions, we can improve this to $1/4-\varepsilon$ for 2-Max-ATSP and $3/8-\varepsilon$ for 2-Max-STSP. The latter is an improvement over the existing deterministic 7/27 approximation for 2-Max-STSP [13,16].

2. Max-ATSP

The rough idea behind our algorithm for k-Max-ATSP is as follows: First, we "guess" a few edges that we contract to get a slightly smaller instance. The number of edges that we have to contract depends only on k and ε. Second, we compute k maximum-weight matchings in the smaller instance, each with respect to one of the k objective functions. Third, we