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Note

On graphs which can or cannot induce Chinese Postman games with a non-empty core[☆]

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ABSTRACT

We study the Chinese postman (CP) cooperative game induced by a connected, weighted, undirected graph G, wherein players reside at the edges of G and a postman, starting from a post-office location (i.e., a vertex of G), needs to traverse all edges before returning to the post-office. We provide a complete characterization of all connected graphs for which there exists a positive edge-cost function such that the induced CP game has a non-empty core, and, consequently, we derive a complete characterization of all connected graphs for which there does not exist a positive edge-cost function which induces a CP game with a non-empty core. Membership in these classes of graphs can be verified in strongly polynomial time.

1. Introduction

In the Chinese postman problem (e.g., [8,2]), defined on a weighted undirected connected graph G, one seeks a least weighted tour which starts at some vertex v_0 of G, traverses all the edges in G at least once, and returns to v_0. Hamers et al. [7] formulated and analyzed a cost allocation problem associated with the Chinese postman problem described as follows: A postman is located at some fixed vertex of a graph G, to be referred to as the *post-office*, and each edge of G belongs to a different player. The players need some service, e.g., mail delivery, and the nature of this service requires the postman to travel from the post-office and visit all edges in the graph, before returning to the post-office. The objective in the cost allocation problem is to find a fair allocation of the cost of a cheapest tour, which starts at the post-office, visits each edge of G at least once, and returns to the post office. Hamers et al. [7] modeled this cost allocation problem as a cooperative game, referred to as the *Chinese postman (CP) game*, in which the cost of each coalition is the lowest cost that the postman incurs in a tour that starts at the post office, traverses all edges in which members of the coalition reside, before returning to the post office. Then, following what is already a well established approach, see, e.g., [1,7] have proceeded to investigate the possibility of using the core, the most important solution concept in cooperative games, in order to generate cost allocation schemes to the above delivery problem. In general, a CP game associated with an undirected connected graph could have an empty core. However, Hamers et al. [7] have shown that a CP game induced by a connected weakly Eulerian graph has a non-empty core. Here, an undirected and connected graph G is called *Eulerian* if the degrees of all its nodes are even, and it is called *weakly Eulerian* if it consists of Eulerian components connected in a tree-like structure. Further, Hamers [6] has shown that if a connected undirected graph is *weakly cyclic*, namely, if every edge therein is contained in at most one cycle, then the associated CP game is *convex*. The convexity of a game implies additional attractive properties for its core, such as the Shapley value is the barycentre of the core [10], and the Aumann–Davis–Maschler bargaining set coincides with the core [9].

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It turns out that the class of weakly Eulerian graphs is the largest class of graphs which always induce CP games with a non-empty core. Specifically, define a graph to be *CP-balanced*, if the corresponding CP game has a non-empty core for all nonnegative edge costs and all locations of the post-office. Then, it was shown by Granot et al. [5] that an undirected graph is CP-balanced if and only if it is weakly Eulerian. This result was strengthened by Granot and Hamers [3], who have proved that an undirected graph is CP balanced, or, equivalently, weakly Eulerian if and only if the corresponding CP game has a non-empty core for all non-negative edge-costs and only for some location of the post-office. Finally, Granot et al. [4] have characterized the class, Λ, of all graphs such that for all $G \in \Lambda$, in the core of the induced CP game corresponding to G, players on each road in G pay exactly the cost of the road, for all locations of the post-office and for all non-negative edge-costs. Here a *road* in G is a maximal path, all of whose interior vertices have a degree equal to two therein. Further, for the class of graphs Λ, which is a proper subset of the class of Eulerian graphs, it was shown by Granot et al. [4] that the core of a CP game corresponding to $G \in \Lambda$ is the Cartesian product of the cores of the CP games induced by the roads in G, and that core membership for a CP game induced by $G \in \Lambda$ can be verified in linear time in the number of edges.

A natural question that can be raised, in view of the above results, is whether a non-weakly Eulerian graph G can induce a CP game whose core is not empty. Clearly, by Granot et al. [5] and Granot and Hamers [3], such a graph G cannot induce a CP game with a non-empty core for all non-negative edge-costs. But, can it induce a CP game with a non-empty core for some positive edge-costs? Indeed, we provide in this paper a complete characterization of the class of all undirected connected graphs, to be referred to as the class of *core-nonempty CP graphs*, which properly subsumes the class of weakly Eulerian graphs, such that each graph in this class induces a CP game with a non-empty core for some positive edge-costs and a fixed location of the post-office.

As importantly, our complete characterization of the class of core-nonempty CP graphs provides us as well with a complete characterization of all connected undirected graphs which cannot induce a CP game whose core is non-empty for any positive edge-costs and for a fixed location of the post-office. We will refer to this class as the class of *core-empty* CP graphs, and we note that it logically complements the class of weakly Eulerian graphs in the sense that the latter class provides the description of all graphs whose topology alone is sufficient to determine non-emptiness of the core of the induced CP game while the former class provides a complete description of all graphs whose topology alone is sufficient to determine core-emptiness of the induced CP game. Indeed, our results complete the search for graphs whose topology alone is sufficient to determine emptiness/non-emptiness of the core of the induced CP game. Finally, we demonstrate that one can recognize in strongly polynomial time (in the number of edges), core-empty or core-nonempty CP graphs.

2. Preliminaries and notation

Let $G = (V(G), E(G))$ be a connected undirected graph with a vertex set $V(G)$, edge set $E(G)$, a special vertex called *post-office*, denoted v_0, and a non-negative edge-cost function ℓ defined over $E(G)$. The players reside at $E(G)$ such that precisely one player is residing at each edge. Therefore, we may identify the set of players $N(G)$ with the set of edges of $E(G)$. That is, $N(G) = E(G)$.

The *postman* (not considered a player) is supposed to start at the post office and deliver service (e.g., mail) to all the players before returning to the post-office. The issue is how to share his cost among the players.

Definition 2.1. A *walk* in a graph G is a sequence $w := v_1, e_1, v_2, \ldots, e_k, v_{k+1}$, where $k \geq 1, v_1, \ldots, v_{k+1} \in V(G); e_1, \ldots, e_k \in E(G)$, such that $e_j = (v_j, v_{j+1})$ for all $j \in \{1, \ldots, k\}$. The vertices v_1 and v_{k+1} are called the *extreme vertices*. If they coincide, the walk is said to be *closed*. If all edges are distinct then the (closed) walk is a (closed) *path*. G is said to be *connected* if there exists at least one path between any two distinct vertices in G. An *edge cutset* in a connected graph $G = (V(G), E(G))$ is a set of edges $A, A \subseteq E(G)$, whose removal disconnets G. An edge cutset in G is said to be *minimal* if no proper subset thereof is also an edge cutset in G.

A *tour* in a graph G, with a post-office node v_0, is defined as a closed walk originating and terminating at v_0. A tour T in G is said to be feasible for coalition S, if T traverses every edge in which a player in S resides. For a weighted graph $G = (V(G), E(G))$, with edge-cost function ℓ, we denote by $k(G)$ the total cost of all edges in G. That is, $k(G) = \sum(\ell_j : j \in E(G))$. Similarly, for a tour T in G, $T = v_0, e_1, v_1, e_2, \ldots, e_k, v_0$, $k(T) = \sum(\ell_{e_j} : j = 1, \ldots, k)$, recognizing that some of the edges in T are not distinct.

Definition 2.2. The Chinese Postman (CP) game, $(N(G), c_{\ell_G})$, induced by a connected weighted graph G, a post-office node v_0, and an edge-cost function ℓ is defined by

$$c_{\ell_G}(S) = \min\{k(T) : T \text{is a feasible tour for } S\}, \quad \forall S \subseteq N(G).$$

The core, $C[(N, c)]$, of a cooperative game (N, c) is defined as the set of all cost allocations for which no subset of players has an incentive to secede from the grand coalition and act on their own. Specifically, $C[(N, c)] = \{x : x \in R^N, x(S) \leq c(S),$ for each $S \subseteq N, x(N) = c(N)\}$, where $x(S) \equiv \Sigma_{j \in S} x_j$.

For the rest of this paper we restrict attention to graphs G with positive edge-cost functions ℓ. That is, $\ell : E(G) \to R_{++}$.