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Finitary and cofinitary gammoids

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ABSTRACT

A gammoid is a matroid defined using linkability of vertex sets in a (possibly infinite) digraph. Related types of matroids are strict gammoids and transversal matroids, three aspects of which will be considered as follows.

First, we investigate the interaction between matroid properties of strict gammoids and transversal matroids and graphic properties of the defining graphs. In particular, we characterize cofinitary strict gammoids and cofinitary transversal matroids among, respectively, strict gammoids and transversal matroids in terms of the defining graphs.

The set of finite circuits of a matroid defines the finitarization matroid on the same ground set. A matroid is nearly finitary if every base can be extended to a base of the finitarization matroid by adding finitely many elements. Aigner-Horev et al. (2011) [6] raised the question whether the number of such additions is bounded for a fixed nearly finitary matroid. We answer this question positively in the classes of strict gammoids and transversal matroids.

Piff and Welsh (1970) proved the classical result that a finite strict gammoid/transversal matroid is representable over any large enough field. In this direction, we prove that finitary strict gammoids and transversal matroids are representable over some field, and hence the cofinitary counterparts are thin sums representable, a new notion of representability introduced by Bruhn and Diestel (2011).

1. Introduction

Transversal matroids and gammoids are defined via the concepts of matchability and linkability in, respectively, bipartite graphs and digraphs (see Edmonds and Fulkerson [14], Mason [19] and Perfect [24]). In case the defining graphs are finite, a matroid structure can be endowed on the linkable sets or matchable sets. In an infinite graph, these concepts can be studied via the notion of infinite matroids. Classically, infinite matroids are finitary, in the sense that a set is independent if and only if all its finite subsets are independent. However, with this definition, the dual of a matroid, for example a uniform matroid, need not be a matroid. In response to this, there has been a variety of proposals to define infinite matroids allowing for duality (for more on the history, see Higgs [16], Oxley [21,22] and Bruhn et al. [11]).

Recently, simple axiomatizations of infinite matroids that extend transparently the rich finite matroid theory were given by Bruhn et al. [11]. Since then, there has been an ongoing project ([4,3] and [12]) which focuses on infinite gammoids and transversal matroids. Here we contribute in three ways: by giving graph theoretic characterizations of cofinitary strict gammoids and transversal matroids, proving the existence of certain bounds on extensions of bases when these types of matroids are nearly finitary, as well as showing that any strict gammoid or transversal matroid that is finitary or cofinitary is thin sums representable.

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Independent sets of a transversal matroid are the matchable subsets of a fixed, say left, vertex class in a bipartite graph; while those of a strict gammoid are the sets linkable to a fixed set of sinks of a digraph. As in [4], an ordered pair of a digraph and a fixed set of sinks will be called a dimaze, where these sinks are called exits. As the bipartite graph and the dimaze influence the corresponding matroids strongly, it is of interest to study how graph properties, which are often easy to visualize, force certain matroid properties. For example, consider finitary transversal matroids. A simple compactness proof (see for example [20]) shows that a bipartite graph, with every vertex in the left vertex class of finite degree, defines a finitary transversal matroid. This condition on the bipartite graph in fact characterizes among transversal matroids the finitary ones, namely, a transversal matroid is finitary if and only if it can be defined via a bipartite graph all of whose vertices in the ground set have finite degree (Proposition 3.2). In [12], Carmesin gave a similar characterization for finitary strict gammoids (Theorem 3.3) in terms of forbidden families of digraphs in a defining dimaze.

We first focus on the classes of cofinitary transversal matroids and cofinitary strict gammoids. It is not hard to see that if we have a strict gammoid defined on a dimaze, then the set of vertices linkable to a fixed exit constitutes a cocircuit, as this is a minimal set meeting every base. Thus, any dimaze defining a cofinitary strict gammoid cannot have, for example, an infinite directed ray arriving at an exit. By analysing more precisely structures that lead to infinite cocircuits, as well as adapting a conversion between matchings and linkages of Ingleton and Piff [17], we prove characterizations of cofinitary transversal matroids (Theorem 3.8) and cofinitary strict gammoids (Theorem 3.10).

Our second topic concerns nearly finitary matroids, which were introduced in [6] to encompass a class larger than finitary matroids, in which one has a matroid union theorem. The finitarization of a matroid is defined by declaring a set independent as soon as all its finite subsets are independent in the given matroid. A matroid is nearly finitary if we can only add finitely many elements to any base while preserving independence in the finitarization. In general, the number of elements added may vary across bases (see Section 4 for an example). An open problem is whether there is a bound on the number of element additions over all bases of a given nearly finitary matroid. We prove the existence of such a bound for nearly finitary strict gammoids and nearly finitary transversal matroids in Propositions 4.2 and 4.8.

Traditionally, representable matroids are defined by linear independence in a family of vectors of a vector space and are necessarily finitary. It was proved by Lindström [18], and Piff and Welsh [25], that any given finite gammoid is representable over any large enough field. Infinite gammoids need not be representable anymore as not every infinite gammoid is finitary. To provide a means to represent (possibly non-finitary) infinite matroids, Bruhn and Diestel [10] introduced the notion of thin sums representability, which was later proved to generalize representability by the first author and Bowler [2]. Our investigation along this line leads us to a transversal matroid and a strict gammoid that are not thin sums representable. However, we show that if a strict gammoid or a transversal matroid is finitary or cofinitary, then it is thin sums representable (Proposition 5.1 and Corollary 5.2).

2. Preliminaries

We collect definitions and notations. For those not found here, we refer the reader to [11] and [23] for matroid theory, and [13] for graph theory.

Analogous to finite matroids, infinite matroids can be axiomatized in different terms, including independent sets, circuits, bases, rank function, and closure operator [11]. The main difference from the finite case is that all of the infinite axiomatizations contain a central axiom which demands for the existence of certain maximal sets. In this paper, we find it convenient to work with the independence axioms.

Given a set E and $\mathcal{I} \subseteq 2^E$, let $\mathcal{I}^{\max}$ denote the maximal elements of $\mathcal{I}$ with respect to set inclusion. For sets I and $\{x\}$, $I + x$ stands for $I \cup \{x\}$. A *matroid* M is a pair $(E, \mathcal{I})$, which satisfies the following:

(I1) $\emptyset \in \mathcal{I}$;
(I2) if $I \subseteq I'$ and $I' \in \mathcal{I}$ then $I \in \mathcal{I}$;
(I3) for every $I' \in \mathcal{I}^{\max}$ and $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$; and
(IM) whenever $I \in \mathcal{I}$ and $I \subseteq X \subseteq E$, the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

We call E the *ground set* of M. A subset of E is *independent* if it is in $\mathcal{I}$, *dependent* otherwise. As usual, we will identify M with its set of independent sets. *Bases* are the maximal independent sets and *circuits* are the minimal dependent sets. The *dual* of M is a matroid M^* on the same ground set whose bases are the complements of the bases of M. Given a set $X \subseteq E$, a *deletion minor* or a *restriction* $M \setminus X = M|(E \setminus X)$ is a matroid on $E \setminus X$ whose independent sets are $\mathcal{I} \cap 2^{E\setminus X}$. The *contraction* of M to X is defined as $(M^*|X)^*$ and denoted by $M.X = M/(E \setminus X)$.

Given a set system $M = (E, \mathcal{I})$, let $\mathcal{I}(M^{\text{fin}}) := \{I \subseteq E : \text{all finite subsets of } I \text{ are in } \mathcal{I}(M)\}$. Then the *finitarization* M^{fin} of M is defined as $(E, \mathcal{I}(M^{\text{fin}}))$. It follows that any minimal dependent set of M^{fin} is a finite minimal dependent set of M. We say that M is *finitary* if and only if $M = M^{\text{fin}}$.

We now turn to one of our main objects which are set systems defined via systems of paths in digraphs equipped with a distinguished set of vertices. Let $D = (V, E)$ be a digraph and $B_0 \subseteq V$ a set of sinks. Call the pair (D, B_0) a *dimaze*, which is an abbreviation for *di*rected *maze*, and B_0 the (set of) *exits*. A *linkage* is a set of vertex disjoint paths ending in B_0. A subset of V is *linkable* if there is a linkage whose set of initial vertices includes the given set. The pair of V and the set of linkable subsets of V is denoted by $M_L(D, B_0)$. If $M_L(D, B_0)$ is a matroid, it is called a *strict gammoid* and (D, B_0) is called a *presentation*