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# A lower bound on the crossing number of uniform hypergraphs

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ABSTRACT

In this paper, we consider the embedding of a complete $d$-uniform geometric hypergraph with $n$ vertices in general position in $\mathbb{R}^d$, where each hyperedge is represented as a $(d-1)$-simplex, and a pair of hyperedges is defined to cross if they are vertex-disjoint and contain a common point in the relative interiors of the simplices corresponding to them. As a corollary of the Van Kampen–Flores Theorem, it can be seen that such a hypergraph contains $\Omega(\frac{2^d}{\sqrt{d}})\binom{n}{2d}$ crossing pairs of hyperedges. Using Gale Transform and Ham Sandwich Theorem, we improve this lower bound to $\Omega(\frac{2^d \log d}{\sqrt{d}})\binom{n}{2d}$.


## 1. Introduction

Hypergraphs are natural generalization of graphs. A hypergraph is a pair $(V, E)$ where $V$ is a finite set and $E \subseteq 2^V$ is a collection of subsets of $V$ [8]. The elements of $E$ are called hyperedges. Given $n$ points in general position in $\mathbb{R}^d$, a geometric $(i+1)$-uniform hypergraph is defined as a collection of $i$-dimensional simplices as hyperedges, induced by some $(i+1)$-tuples from the point set [4]. In this paper, we consider $i = d-1$. A complete geometric $d$-uniform hypergraph on $n$ vertices is denoted as $K_n^d$ in this paper. The case $d = 2$, $i = 1$, known as the rectilinear crossing number problem of a graph, has been studied in detail in the literature [1–3,6]. In a *rectilinear drawing* of a graph $G$ in the plane, vertices are represented by points in a general position (i.e., no three points lie on a line) and edges are represented by straight line segments connecting the corresponding points. The *rectilinear crossing number* of a graph $G$ is the minimum number of crossing pairs of edges over all rectilinear drawings of $G$.

For $d \geq 2$, we define the *d-dimensional rectilinear drawing* of a $d$-uniform hypergraph $H$ as follows. The vertices of $H$ are represented by points in a general position in $\mathbb{R}^d$ (i.e., no $d+1$ of these points lie on a hyperplane) and each hyperedge is represented as the convex hull of the points corresponding to its vertices. A pair of hyperedges *overlaps* if they have a common point in the relative interiors of the simplices corresponding to them. It can be easily seen that a pair of 2-simplices in $\mathbb{R}^3$ can overlap in two different ways. The first way, as shown in Fig. 1, is called a *crossing* and the second way is called an *intersection*. Similarly in $\mathbb{R}^d$, there are various ways in which a pair of hyperedges can overlap. We call a pair of overlapping hyperedges to be *crossing* if they have no vertices in common. The *d-dimensional rectilinear crossing number* $\overline{cr}_d(H)$ of a $d$-uniform hypergraph $H$ is defined as the minimum possible number of pairwise crossings of its hyperedges over all $d$-dimensional rectilinear drawings of $H$.

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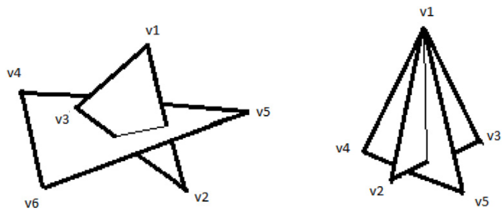


**Fig. 1.** (i) (Left) Crossing between hyperedges $\{v1, v2, v3\}$ and $\{v4, v5, v5\}$. (ii) (Right) Intersection between hyperedges $\{v1, v2, v3\}$ and $\{v1, v4, v5\}$.

As defined earlier, $\overline{cr}_2(K_n^2)$ denotes the minimum number of crossing pairs of edges in a rectilinear drawing of $K_n^2$. The best known lower and upper bounds on this number are $0.37997\binom{n}{4} + \Theta(n^3) \leq \overline{cr}_2(K_n^2) \leq 0.380488\binom{n}{4} + \Theta(n^3)$, due to [1] and [2], respectively. It is quite easy to show that the minimum number of pairwise crossing triangles in a complete geometric 3-uniform hypergraph $K_n^3$ embedded in $\mathbb{R}^3$ is $\binom{n}{6}$. This follows from the fact that any set of 6 vertices in a general position in $\mathbb{R}^3$ contains a crossing pair of hyperedges from $K_n^3$. (See the Geometric Van Kampen–Flores Theorem below.) For a general dimension $d \geq 3$, let us denote by $c_d$ the minimum number of crossing pairs of $(d-1)$-simplices spanned by a set of $2d$ vertices placed in general position in $\mathbb{R}^d$. It follows that $\overline{cr}_d(K_n^d) \geq c_d\binom{n}{2d}$.

In order to obtain a lower bound on $c_d$, we first use the geometric version of Van Kampen–Flores Theorem [5,9].

**Theorem A** (*Geometric Van Kampen–Flores Theorem*)**.** *For any $k \geq 1$, any set of $2k+3$ points in $\mathbb{R}^{2k}$ contains two disjoint subsets A and B such that the convex hulls of A and B have a common point in their relative interiors, and $|A| = |B| = k+1$.*

If $d$ is even, i.e., $d = 2k$ for some $k$, this Theorem shows the existence of a crossing pair of $\frac{d}{2}$-simplices among the $\frac{d}{2}$-simplices formed by any set of $d+3$ vertices selected out of $2d$ vertices placed in general position in $\mathbb{R}^d$. This crossing pair can be extended to crossing pairs of $(d-1)$-simplices in $\binom{d-2}{\frac{d-2}{2}} = \Theta(\frac{2^d}{\sqrt{d}})$ ways. If $d$ is odd, i.e., $d = 2k'-1$ for some $k'$, we map a set of $d+3$ vertices in $\mathbb{R}^d$ to $d+3$ vertices in $\mathbb{R}^{d+1}$ by adding a 0 as the last coordinate of all these vertices. We also add one dummy vertex whose first $d$ coordinates are 0 each, and whose last coordinate is non-zero. By the Geometric Van Kampen–Flores Theorem, this set of $2k'+3$ vertices in $\mathbb{R}^{2k'}$ contains a crossing pair of $(\frac{d+1}{2})$-simplices. Note that neither of these simplices contain the dummy vertex, as it is the only vertex having a coordinate in the $(d+1)$-st dimension and therefore cannot be involved in a crossing. As a result, this crossing pair can be extended to crossing pairs of $(d-1)$-simplices in $\binom{d-3}{\frac{d-3}{2}} = \Theta(\frac{2^d}{\sqrt{d}})$ ways.

### 1.1. Our contribution

In Section 3, we first prove the following result, which implies that $\overline{cr}_4(K_n^4) \geq 4\binom{n}{8}$.

**Theorem 1.** *There are at least 4 crossing pairs of 3-simplices spanned by a set of 8 points in general position in $\mathbb{R}^4$.*

Thereafter, we use a similar idea to show that $c_d = \Omega(\frac{2^d \log d}{\sqrt{d}})$, which implies that $\overline{cr}_d(K_n^d) = \Omega(\frac{2^d \log d}{\sqrt{d}})\binom{n}{2d}$.

**Theorem 2.** *There are $\Omega(\frac{2^d \log d}{\sqrt{d}})$ crossing pairs of $(d-1)$-simplices spanned by a set of $2d$ points in general position in $\mathbb{R}^d$.*

In Section 4, we show the existence of 8 points in general position in $\mathbb{R}^4$ such that there exist 4 crossing pairs of 3-simplices formed by these points. This shows that the bound in Theorem 1 is optimal. However, we do not know whether the bound in Theorem 2 is optimal since we are only aware of the trivial upper bound $\overline{cr}_d(K_n^d) = O(\frac{4^d}{\sqrt{d}})\binom{n}{2d}$, which follows from the fact that the number of distinct $(d-1)$ simplices spanned by a set of $2d$ points in general position in $\mathbb{R}^d$ is $\binom{2d}{d}$.

As far as we know, this is the first non-trivial lower bound obtained on $\overline{cr}_d(K_n^d)$. It is an exciting open problem to find out whether this lower bound is tight.

## 2. Techniques used

### 2.1. Gale transformation

For a positive integer $m$, consider a set $A$ of $m+d+1$ points $x_1, x_2 \ldots x_{m+d+1}$ in general position in $\mathbb{R}^d$. It is easy to see that the null space of the matrix