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A $\frac{5}{4}$-approximation for subcubic 2EC using circulations and obliged edges[☆]

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ABSTRACT

In this paper we study the NP-hard problem of finding a minimum size 2-edge-connected spanning subgraph (henceforth 2EC) in cubic and subcubic multigraphs. We present a new $\frac{5}{4}$-approximation algorithm for 2EC for subcubic bridgeless multigraphs, improving upon the current best approximation ratio of $\frac{5}{4}+\varepsilon$. Our algorithm involves an elegant new method based on circulations which we feel has potential to be more broadly applied. We also study the closely related integrality gap problem, i.e. the worst case ratio between the integer linear program for 2EC and its linear programming relaxation, both theoretically and computationally. We show this gap is at most $\frac{5}{4}$ for subcubic bridgeless multigraphs, and is at most $\frac{9}{8}$ for all subcubic bridgeless graphs with up to 16 nodes. Moreover, we present a family of graphs that demonstrate the integrality gap for 2EC is at least $\frac{8}{7}$, even when restricted to subcubic bridgeless graphs. This represents an improvement over the previous best known bound of $\frac{9}{8}$.

1. Introduction

Given an unweighted bridgeless multigraph $G = (V, E)$, $|V| = n$, the *minimum size 2-edge-connected spanning subgraph problem* (henceforth 2EC(G)) consists of finding a 2-edge-connected spanning subgraph H of G with the minimum number of edges. Note that a *2-edge-connected* (or *bridgeless*) graph $G = (V, E)$ is one that remains connected after the removal of any edge. In a solution for 2EC(G), multiple copies of an edge $e \in E$ are not allowed (and also not necessary, see [3]).

The problem 2EC(G) is one of the most extensively studied problems in network design. It relates to the optimal design of a network that can survive the loss of a link, and thus has many real world applications. However, it is known to be NP-hard and also MAX SNP-hard even for subcubic graphs [4], where a graph is cubic if every node has degree 3, and subcubic if every node has degree at most 3. Thus research has focused on finding good approximation algorithms. Unfortunately, finding improved approximation algorithms seems to be difficult for the more general weighted version of 2EC(G) where, as with the closely related traveling salesman problem (TSP), the best known approximation ratio for metric weights has remained at $\frac{3}{2}$ [1] without any improvement for over 30 years.

Given the difficulty of this problem, people have turned to the study of approximation algorithms for special cases, which has proven to be a more successful approach for 2EC(G) than studying its more general weighted form. In such studies, not only improved results were obtained but also new innovative methods which may lead to more general results.

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In this paper we focus on the simplest form of 2EC(G) that still remains NP-hard, i.e. 2EC(G) for subcubic graphs. In Section 2 we describe the framework for a new innovative method for designing approximation algorithms for 2EC(G) based on circulations and the concept of obliged edges, i.e. edges that must be included in the final solution. Similar types of circulations were used in [10] in the approximation of graph TSP, however the goal was quite different in that context. In fact, to the best of our knowledge, circulations have not previously been used in the way we describe to approximate 2EC(G). In Section 3 we develop a new heuristic method for the problem 2EC(G) for bridgeless cubic graphs with obliged edges. In Section 4 we demonstrate the usefulness of this method by using it to develop a new $\frac{5}{4}$-approximation algorithm for 2EC(G) on bridgeless subcubic multigraphs. This algorithm improves upon the previous best approximation ratio of $\frac{5}{4}+\varepsilon$ given by Csaba, Karpinski and Krysta [4] for 2EC(G) on such graphs. We feel this algorithm not only provides a modest improvement in the approximation ratio, but also, and perhaps more importantly, provides an improvement in the simplicity of the method and proof.

A related approach for finding approximated 2EC(G) solutions is to study the integrality gap $\alpha^{2\text{EC}(G)}$, which is the worst case ratio between the optimal value for 2EC(G) and the optimal value for its linear programming (henceforth LP) relaxation (see [1] for background). As a critical topic throughout this paper, we study $\alpha^{2\text{EC}(G)}$ intensively. There are two main reasons this is useful. First, the integrality gap itself serves as an indicator of the quality of the lower bound given by the LP relaxation. This is important for methods, such as branch and bound and approximation, that depend on good lower bounds for their success. Secondly, an algorithmic proof for $\alpha^{2\text{EC}(G)} = k$ yields a k-approximation algorithm for 2EC(G) [1]. In this paper, we give an upper bound on the value of $\alpha^{2\text{EC}(G)}$ on bridgeless cubic graphs with an algorithmic proof, while lower bounds on the integrality gap of 2EC(G) are investigated through computational studies. In Section 4, we show that the integrality gap of 2EC(G) is strictly less than $\frac{5}{4}$ for bridgeless subcubic multigraphs, improving on the previous best known bound of $\frac{5}{4}+\varepsilon$ [4]. In Section 5 we describe a computational study we conducted by designing a program that calculates $\alpha^{2\text{EC}(G)}$ exactly for all simple graphs $G \in \mathcal{G}$, where $\mathcal{G}$ contains all test cases in three categories: (1) General bridgeless graphs for $3 \le n \le 10$; (2) Cubic bridgeless graphs for $6 \le n \le 16$; (3) Subcubic bridgeless graphs for $3 \le n \le 16$. Using the knowledge gained through the data analysis for the computational study, we obtain a family of subcubic bridgeless graphs G which shows $\alpha^{2\text{EC}(G)} \ge \frac{8}{7}$ asymptotically, providing an improvement upon the previous best lower bound of $\frac{9}{8}$ [12].

1.1. *Literature review on* 2EC(G)

Constant factor approximation algorithms for 2EC(G) have been intensely studied. In 1994, Khuller and Vishkin [7] found a $\frac{3}{2}$-approximation, which was improved by Cheriyan, Sebő and Szigeti [3] to $\frac{17}{12}$. The ratio was later improved to $\frac{4}{3}$ in 2000 by Vempala and Vetta [14]. One year later, Krysta and Kumar [8] improved the approximation ratio to $\frac{4}{3}-\varepsilon$ where $\varepsilon = \frac{1}{1344}$. Recently, Sebő and Vygen [12] designed a simpler and more elegant $\frac{4}{3}$-approximation algorithm for 2EC(G).

In the meantime, research on 2EC(G) has also been conducted for special classes of graphs, especially on cubic and subcubic bridgeless graphs, on which 2EC(G) still remains NP-hard [4]. In 2001, along with their $(\frac{4}{3}-\varepsilon)$-approximation algorithm for 2EC(G) on general graphs, Krysta and Kumar [8] also presented an approximation algorithm for 2EC(G) on cubic graphs with the approximation ratio of $\frac{21}{16}+\varepsilon$. One year later, Csaba, Karpinski and Krysta [4] designed a $(\frac{5}{4}+\varepsilon)$-approximation algorithm for 2EC(G) on subcubic bridgeless graphs. In 2004, Huh [6] presented an algorithm yielding a $\frac{5}{4}$-approximation on cubic 3-edge-connected graphs. A more recent improvement came from Boyd, Iwata and Takazawa [2] with a $\frac{6}{5}$-approximation algorithm for 2EC(G) on cubic 3-edge-connected graphs.

Concerning the integrality gap $\alpha^{2\text{EC}(G)}$ of 2EC(G), Csaba, Karpinski and Krysta [4] proved that for maximum degree 3 graphs, the integrality gap of the LP relaxation for 2EC(G) is at most $\frac{5}{4}+\varepsilon$ for any fixed $\varepsilon > 0$. It was also stated in [4] that the best known lower bound on $\alpha^{2\text{EC}(G)}$ is $\frac{10}{9}$ for subcubic graphs. In 2013, Boyd, Iwata and Takazawa [2] showed $\alpha^{2\text{EC}(G)} \le \frac{6}{5}$ for 3-edge-connected cubic graphs. Around the same time, Sebő and Vygen [12] proved that $\frac{9}{8} \le \alpha^{2\text{EC}(G)} \le \frac{4}{3}$ in general.

1.2. *Notation and background*

For the purpose of this paper, any graph $G = (V, E)$ is considered to be a multigraph without loops. We use n to denote $|V|$, and sometimes use $V(G)$ to denote V and $E(G)$ to denote E. For any $S \subset V$, $\delta(S)$ is the set of edges with one end in S and the other end not in S, and for any $F \subseteq E$, the notation $x(F)$ is used to denote $\sum_{e\in F} x_e$.

Denoted by $ILP(G)$, the integer linear program of 2EC(G) for a graph $G = (V, E)$ is given as follows:

$$\text{Minimize} \quad \sum_{e\in E} x_e \tag{1}$$

$$\text{subject to} \quad x(\delta(S)) \ge 2 \quad \text{for all } \emptyset \subset S \subset V, \tag{2}$$

$$0 \le x_e \le 1 \quad \text{for all } e \in E, \tag{3}$$

$$x_e \text{ integer} \quad \text{for all } e \in E. \tag{4}$$

By relaxing the integrality constraints (4) of $ILP(G)$, the LP relaxation of $ILP(G)$, denoted by $LP(G)$, is obtained. We use the notation $OPT(G)$ and $OPT_{LP}(G)$ to denote the optimal objective value for $ILP(G)$ and $LP(G)$ respectively.