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A lower bound on the independence number of a graph in terms of degrees and local clique sizes

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ABSTRACT

Caro and Wei independently showed that the independence number $\alpha(G)$ of a graph G is at least $\sum_{u\in V(G)}\frac{1}{d_G(u)+1}$. In the present paper we conjecture the stronger bound $\alpha(G)\geq\sum_{u\in V(G)}\frac{2}{d_G(u)+\omega_G(u)+1}$ where $\omega_G(u)$ is the maximum order of a clique of G that contains the vertex u. We discuss the relation of our conjecture to recent conjectures and results concerning the independence number and the chromatic number. Furthermore, we prove our conjecture for perfect graphs and for graphs of maximum degree at most 4.

1. Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation. Independence in graphs is one of the most well studied topics of graph theory. Since even the approximation of the independence number $\alpha(G)$ of a graph G is algorithmically very hard [10], lower bounds on the independence number have received much attention. Caro [5] and Wei [16] independently proved the following classical lower bound.

Theorem 1.1 (*Caro [5], Wei [16]*). *If G is a graph, then*

$$\alpha(G)\geq\sum_{u\in V(G)}\frac{1}{d_G(u)+1}.$$

For a graph G of order $n(G)$ and maximum degree $\Delta(G)$, this result immediately implies $\alpha(G)\geq\frac{n(G)}{\Delta(G)+1}$, which, by Brooks' theorem [4], is best-possible for cliques and odd cycles. Since the clique number $\omega(G)$ of G satisfies $\omega(G)\leq\Delta(G)+1$, the following result due to Fajtlowicz [6] is a strengthening of the above observation.

Theorem 1.2 (*Fajtlowicz [6]*). *If G is a graph, then*

$$\alpha(G)\geq\frac{2n(G)}{\Delta(G)+\omega(G)+1}.$$

Recently, Henning et al. [11] showed a more localized version of Fajtlowicz's bound.

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Theorem 1.3 (*Henning et al. [11]*)**.** *Let G be a graph. If p is an integer such that for every clique C of G, there is a vertex u in C with $d_G(u) + |C| + 1 \leq p$, then*

$$\alpha(G) \geq \frac{2n(G)}{p}.$$

The minimum value of p satisfying the condition in Theorem 1.3 can be smaller than $\Delta(G) + \omega(G) + 1$ because of its more local definition.

The following alternative local refinement of Fajtlowicz's bound was conjectured by Bertram and Horak in [1].

Conjecture 1.4 (*Bertram and Horak [1]*)**.** *If G is a graph, then*

$$\alpha(G) \geq \sum_{u \in V(G)} \frac{2}{d_G(u) + \omega(G) + 1}.$$

In [1] Conjecture 1.4 was established for regular graphs as well as for triangle-free graphs. Further strengthenings and versions of Caro's and Wei's bound were considered in [2,3,8,9].

In the present paper we propose the following strengthening of Conjecture 1.4, where $\omega_G(u)$ denotes the maximum size of a clique of a graph G that contains a specified vertex u of G.

Conjecture 1.5. *If G is a graph, then*

$$\alpha(G) \geq \sum_{u \in V(G)} \frac{2}{d_G(u) + \omega_G(u) + 1}.$$

As our main contributions we show that Conjecture 1.5 holds for graphs of maximum degree at most 4 and for perfect graphs. Note that the bounds in Conjectures 1.4 and 1.5 coincide if every vertex of the considered graph lies in a maximum clique. Using this observation it follows easily from the result in [1] that Conjecture 1.5 holds for triangle-free graphs.

Before we proceed to our results, we explain some connections to the chromatic number $\chi(G)$ and the fractional chromatic number $\chi_f(G)$ of a graph G. Since $\alpha(G) \geq \frac{n(G)}{\chi_f(G)} \geq \frac{n(G)}{\chi(G)}$ for every graph G, upper bounds on the (fractional) chromatic number always imply lower bounds on the independence number as their immediate corollaries. In fact, Fajtlowicz's Theorem 1.2 is an immediate consequence of the following result.

Theorem 1.6 (*Molloy and Reed [14]*)**.** *If G is a graph, then*

$$\chi_f(G) \leq \frac{1}{2}\left(\Delta(G) + \omega(G) + 1\right).$$

This last result is the fractional version of Reed's [15] famous conjecture.

Conjecture 1.7 (*Reed [15]*)**.** *If G is a graph, then*

$$\chi(G) \leq \left\lceil \frac{1}{2}\left(\Delta(G) + \omega(G) + 1\right) \right\rceil.$$

In his thesis, King [12] considered a local strengthening of Reed's conjecture.

Conjecture 1.8 (*King [12]*)**.** *If G is a graph, then*

$$\chi(G) \leq \max_{u \in V(G)} \left\lceil \frac{1}{2}\left(d_G(u) + \omega_G(u) + 1\right) \right\rceil.$$

As observed by McDiarmid, the fractional version of King's conjecture is true.

Theorem 1.9 (*McDiarmid, cf. Theorem 2.10 in [12]*)**.** *If G is a graph, then*

$$\chi_f(G) \leq \max_{u \in V(G)} \frac{1}{2}\left(d_G(u) + \omega_G(u) + 1\right).$$

This implies the following, which yields further support for our Conjecture 1.5.