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Note on the upper bound of the rainbow index of a graph☆

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ABSTRACT

A path in an edge-colored graph G is a rainbow path if every two edges of it receive distinct colors. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the minimum number of colors that are needed to color the edges of G such that there is a rainbow path connecting every two vertices of G. Similarly, a tree in G is a rainbow tree if no two edges of it receive the same color. The minimum number of colors that are needed in an edge-coloring of G such that there is a rainbow tree containing all the vertices of S (other vertices may also be included in the tree) for each k-subset S of $V(G)$ is called the k-rainbow index of G, denoted by $rx_k(G)$, where k is an integer such that $2 \leq k \leq n$. Chakraborty et al. got the following result: For every $\epsilon > 0$, a connected graph with minimum degree at least ϵn has bounded rainbow connection number, where the bound depends only on ϵ. Krivelevich and Yuster proved that if G has n vertices and the minimum degree $\delta(G)$ then $rc(G) < 20n/\delta(G)$. This bound was later improved to $3n/(\delta(G)+1)+3$ by Chandran et al. Since $rc(G) = rx_2(G)$, a natural problem arises: for a general k determining the true behavior of $rx_k(G)$ as a function of the minimum degree $\delta(G)$. In this paper, we give upper bounds of $rx_k(G)$ in terms of the minimum degree $\delta(G)$ in different ways, namely, via Szemerédi's Regularity Lemma, connected 2-step dominating sets, connected $(k-1)$-dominating sets and k-dominating sets of G.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [1]. Let $G = (V, E)$ be a nontrivial connected graph with an edge-coloring $c : E \to \{1, 2, \ldots, \ell\}$, $\ell \in \mathbb{N}$, where adjacent edges may be colored with one same color. A path of G is a *rainbow path* if no two edges of the path are colored with a same color. The graph G is *rainbow connected* if for every two vertices of G, there is a rainbow path connecting them. The minimum number of colors for which there is an edge-coloring of G such that G is rainbow connected is called the *rainbow connection number*, denoted by $rc(G)$. These concepts were introduced by Chartrand et al. in [8]. Since it is almost impossible to give the precise value of the rainbow connection number for an arbitrary graph, many bounds for the rainbow connection number have been given in terms of other graph parameters, such as minimum degree and connectivity, etc. The interested readers can see [4,8,11–13].

In [9], Chartrand et al. generalized the concept of rainbow path to rainbow tree. A tree T in G is a *rainbow tree* if no two edges of T receive the same color. For $S \subseteq V$, a *rainbow S-tree* is a rainbow tree containing all the vertices of S (other vertices may also be included in the tree). Given a fixed integer k with $2 \leq k \leq n$, an edge-coloring c of G is called a *k-rainbow coloring* if for every set S of k vertices in G, there exists a rainbow S-tree. In this case, we called *Gk-rainbow connected*. The minimum

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number of colors that are needed in a *k-rainbow coloring* of G is called the *k-rainbow index*, denoted by $rx_k(G)$. Clearly, when $k = 2$, $rx_2(G)$ is exactly the rainbow connection number $rc(G)$. For every connected graph G of order n, it is easy to see that $rc(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$. We refer to [2,3,10,14] for more details about the k-rainbow index.

Not surprisingly, as the minimum degree increases, the graph would become more dense and therefore the rainbow connection number and rainbow index would decrease. In [6], [11] and [7], the authors studied the relationship between the minimum degree $\delta(G)$ and the rainbow connection number $rc(G)$:

Theorem 1 ([6]). *For every $\epsilon > 0$, a connected graph with minimum degree at least ϵn has bounded rainbow connection number, where the bound depends only on ϵ.*

Theorem 2 ([11]). *If G has n vertices and the minimum degree $\delta(G)$ then $rc(G) < 20n/\delta(G)$.*

Theorem 3 ([7]). *For every connected graph G of order n and minimum degree δ, $rc(G) \leq 3n/(\delta+1)+3$. Moreover, for every $\delta \geq 2$, there exist infinitely many graphs G such that $rc(G) \geq 3(n-2)/(\delta+1)-1$.*

Since $rc(G)$ is the case of $rx_k(G)$ for $k = 2$, a natural problem arises: for a general k determining the true behavior of $rx_k(G)$ as a function of the minimum degree $\delta(G)$. In this paper, we focus on this problem and obtain some upper bounds for $rx_k(G)$ in terms of $\delta(G)$ in different ways, namely, via Szemerédi's Regularity Lemma, connected 2-step dominating sets, connected $(k-1)$-dominating sets and k-dominating sets of G. The main idea is similar to those of [6,11,7]. However, the proofs have their technical details and the results are meaningful.

2. Preliminaries

For a graph G, we use $V(G)$, $E(G)$ and $\delta(G)$ to denote its vertex set, edge set and minimum degree, respectively. For $D \subseteq V(G)$, let $\overline{D} = V(G) \setminus D$, $|D|$ be the number of vertices in D, and $G[D]$ be the subgraph of G induced by D. For two nonempty disjoint vertex subsets X and Y of a graph G, let $E(X, Y)$ denote the set of edges of G between X and Y.

Definition 1. The distance between two vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest path between them in G. The diameter of G, denoted by $diam(G)$, is the maximum distance between every pair of vertices in G. The distance between a vertex v and a set $D \subseteq V(G)$ is $d(v, D) := \min\{d(v, u) : u \in D\}$. For a positive integer k, the k-step neighborhood of a set $D \subseteq V(G)$ is $N^k(D) := \{x \in V(G) : d(x, D) = k\}$. The distance between two sets $X, Y \subseteq V(G)$ is $d(X, Y) := \min\{d(x, y) : x \in X, y \in Y\}$.

Definition 2. The *Steiner distance* $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S. Such a tree is called a *Steiner S-tree* or simply a *Steiner tree*. The *k-Steiner diameter* $sdiam_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G.

It is easy to get a simple upper bound and lower bound for $rx_k(G)$.

Observation 1 ([9]). For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $k-1 \leq sdiam_k(G) \leq rx_k(G) \leq n-1$.

Definition 3. Given a graph G and a positive integer k, a set $D \subseteq V(G)$ is called a *k-step dominating set* of G, if every vertex in G is at a distance at most k from D, i.e., $V(G) = D \cup (\cup_{i=1}^{k} N^i(D))$. Further, if $G[D]$ is connected, we call D a *connected k-step dominating set* of G. The *connected k-step domination number* $\gamma_c^k(G)$ is the number of vertices in a minimum connected k-step dominating set of G. When $k = 1$, we may omit the qualifier "1-step" in the above names and the superscript 1 in the notation.

Definition 4. Given a graph G and a positive integer k, a set $D \subseteq V(G)$ is called a *k-dominating set* of G, if every vertex in $\overline{D}$ is adjacent to at least k distinct vertices of D. Furthermore, if $G[D]$ is connected, we call D a *connected k-dominating set*. The connected k-domination number $\gamma_k^c(G)$ is the minimum cardinality among all the connected k-dominating sets of G.

Definition 5. Given a graph G and a positive integer k, a dominating set D of G is called a *k-way dominating set* if $d(v) \geq k$ for every vertex $v \in \overline{D}$. In addition, if $G[D]$ is connected, we call D a *connected k-way dominating set*.

Definition 6. Let G be a graph and $D \subseteq V(G)$. For $v \in N^1(D)$, its neighbors in D are called *feet* of v, and the corresponding edges are called *legs* of v.

Definition 7. An edge-colored graph is *rainbow* if no two edges in the graph share the same color.