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Matroid union — Graphic? Binary? Neither?

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ABSTRACT

There is a conjecture that if the union (also called sum) of graphic matroids is not graphic then it is nonbinary. Some special cases have been proved only, for example if several copies of the same graphic matroid are given. If there are two matroids and the first one can either be represented by a graph with two points, or is the direct sum of a circuit and some loops, then a necessary and sufficient condition is known for the other matroid to ensure the graphicity of the union and the above conjecture holds for these cases. We prove the sufficiency of this condition for the graphicity of the union of two arbitrary graphic matroids. Then we present a weaker necessary condition which is of similar character. Finally we suggest a more general framework of the study of such questions by introducing matroid classes formed by those graphic (or arbitrary) matroids whose union with any graphic (or arbitrary) matroid is graphic (or either graphic or nonbinary).

1. Introduction

Graphic matroids form one of the most significant classes in matroid theory (For terminology and notation in matroid theory the reader is referred to [3,7] and [8].). When introducing matroids, Whitney concentrated on relations to graphs. The definitions of some basic operations like deletion, contraction and direct sum were straightforward generalizations of the respective concepts in graph theory. Most matroid classes, for example those of binary, regular or graphic matroids, are closed with respect to these operations. This is not the case for the union. The union of two graphic matroids can be non-graphic.

The first paper studying the graphicity of the union of graphic matroids was probably that of Lovász and Recski [2], they examined the case if several copies of the same graphic matroid are given.

Another possible approach is to fix a graph G_0 and characterize those graphs G where the union of their cycle matroids $M(G_0) \vee M(G)$ is graphic. (Observe that we may clearly disregard the cases if G_0 consists of loops only, or if it contains coloops.) As a byproduct of some studies on the application of matroids in electric network analysis, this characterization has been performed for the case if G_0 consists of loops and a single circuit of length two only, see the first graph of Fig. 1. (In view of the above observation this is the simplest nontrivial choice of G_0.)

Theorem 1 ([4,6]). *Let A and B be the cycle matroids of the graphs shown in Fig. 1 on ground sets $E_A = \{1, 2, \dots, n\}$ and $E_B = \{1, 2, i, j, k\}$, respectively. Let M be an arbitrary graphic matroid on E_A.*

Then the union $A \vee M$ is graphic if and only if B is not a minor of M with any triplet i, j, k.

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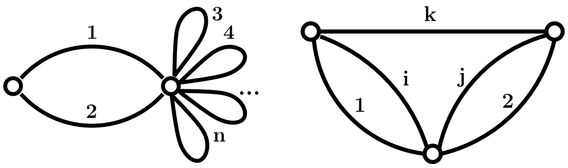

Fig. 1. A graphic representation of A (left) and B (right).

Conjecture 2. *Recski [5] conjectured some thirty years ago that if the union of two graphic matroids is not graphic then it is nonbinary.*

This is known to be true if the two graphic matroids are identical or if one of them is A as given in Theorem 1 — these results follow in a straightforward way from [2] and from [4], respectively.

In a previous paper [1] we extended the result of Theorem 1 if G_0 either consists of loops and two points joined by n parallel edges or if it consists of loops and a single circuit of length n. We proved that deciding whether $M(G_0) \vee M(G)$ is graphic can be performed in polynomial time if G_0 is one of these two matroids. Our results also implied that the above conjecture is true if one of these two types of graphs play the role of G_0.

Observe that the first graph of Fig. 1, representing A, has only two non-loop edges (1 and 2), while the second graph, representing B, has the property that the complement of the set $\{1, 2\}$ of non-loop edges of A contains both a circuit and a spanning tree. This property turned out to be crucial if we consider a larger set of non-loop edges which are either all parallel or all serial.

2. Reduction steps

While during our study of the union of the two graphic matroids $M_1 = M(G_0)$ and $M_2 = M(G)$ the former one had a very special structure in [1], in the present section, we formulate some reduction steps for arbitrary graphic matroids M_1 and M_2 on the same ground set.

Throughout M_1 and M_2 will be graphic matroids on the same ground set E. We shall refer to them as *addends*. It is well known that if a matroid is graphic then so are all of its submatroids and minors. Hence if a matroid has a non-graphic minor then the matroid is not graphic.

Definition 3. We call some non-coloop edges of a matroid serial if they belong to exactly the same circuits.

Definition 4. Let $L(M)$ and $NL(M)$ denote the set of loops and non-loops, respectively, in the matroid M.

The following lemmata contain the main opportunities when we can simplify our addend matroids. Since they refer to graphic matroids only, we can use graph theoretical terminology. Throughout, $M \setminus X$ and M/X will denote deletion and contraction, respectively, of the set X in a matroid M, while $X - Y$ will denote the difference of the sets X and Y. We shall write $Y \cup x$, $Y - x$, $M \setminus x$ and M/x instead of $Y \cup \{x\}$, $Y - \{x\}$, $M \setminus \{x\}$ and $M/\{x\}$, respectively.

2.1. The earlier steps

Lemmata 5 through 11 were proved in [1] and they will be useful for our new results as well.

Lemma 5. *Let X and Y denote the set of coloops in M_1 and in M_2, respectively. The union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic.*

Recall that a matroid is connected if it does not arise as the direct sum of two smaller matroids. If M is not connected and X is the ground set of a connected component of M then $M/X = M \setminus X$.

Lemma 6. *If the ground set of a connected component X of the matroid M_1 is a subset of $L(M_2)$ then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is graphic.*

Recall that the cycle matroid of a loopless graph with no isolated vertices is connected if and only if the graph is 2-vertex-connected.

Lemma 7. *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ in which $X \subset E$ determines a connected subgraph and $E - X$ has exactly two common vertices with X (call them a and b).*

Let M_1' be the cycle matroid of $G' := G(V, (E - X) \cup \{(a, b)\})$ and $M_2' := (M_2 \setminus X) \cup loop(a, b)$ (here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').

If X is a subset of $L(M_2)$ then the union $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.