Discrete Applied Mathematics 209 (2016) 84–91

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the equitable total chromatic number of cubic graphs[☆]

S. Dantas [a], C.M.H. de Figueiredo [b], G. Mazzuoccolo [c], M. Preissmann [d], V.F. dos Santos [e], D. Sasaki [f,*]

[a] IME, Universidade Federal Fluminense, Brazil
[b] COPPE, Universidade Federal do Rio de Janeiro, Brazil
[c] Università di Modena e Reggio Emilia, Italy
[d] CNRS/INPG/UJF-Grenoble 1, G-SCOP UMR5272 Grenoble, F-38031 Grenoble, France
[e] DECOM, Centro Federal de Educação Tecnológica de Minas Gerais, Brazil
[f] PSL, Université Paris-Dauphine and CNRS, LAMSADE UMR 7243, France

ARTICLE INFO

Article history:
Received 14 November 2014
Received in revised form 28 August 2015
Accepted 5 October 2015
Available online 10 November 2015

Keywords:
Cubic graphs
Equitable total-coloring
NP-complete

ABSTRACT

A total coloring is equitable if the number of elements colored with each color differs by at most one, and the least integer for which a graph has such a coloring is called its equitable total chromatic number. Wang conjectured that the equitable total chromatic number of a multigraph of maximum degree Δ is at most $\Delta + 2$, and proved this for the case where $\Delta \leq 3$. Therefore, the equitable total chromatic number of a cubic graph is either 4 or 5, and in this work we prove that the problem of deciding whether it is 4 is NP-complete for bipartite cubic graphs.

Furthermore, we present the first known Type 1 cubic graphs with equitable total chromatic number 5. All of them have, by construction, a small girth. We also find one infinite family of Type 1 cubic graphs of girth 5 having equitable total chromatic number 4. This motivates the following question: Does there exist Type 1 cubic graphs of girth greater than 5 and equitable total chromatic number 5?

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

The Total Coloring Conjecture states that the total chromatic number of any graph is at most $\Delta + 2$, where Δ is the maximum degree of the graph [2]. This conjecture has been proved for cubic graphs, so the total chromatic number of a cubic graph is either 4 (in which case the graph is called *Type* 1) or 5 (*Type* 2) [12,15], see also [6] for a recent concise proof. In 1989, Sánchez-Arroyo [13] showed that the problem of deciding whether the total chromatic number of a bipartite cubic graph is 4 is NP-complete.

The equitable total coloring requires further that the cardinalities of any two color classes differ by at most 1. Similarly to the case of total colorings, it was conjectured that the equitable total chromatic number of any graph is at most $\Delta + 2$ and this conjecture was proved for subcubic graphs in the same work [18]. Furthermore it has even been shown recently

☆ Research was supported by grants from CNPq (Universal 442707/2014-2, PDE 211702/2013-7), CAPES (Math AmSud 021/14) and FAPERJ (Cientistas do nosso Estado E-26/102.952/2011).

* Corresponding author.
E-mail addresses: sdantas@im.uff.br (S. Dantas), celina@cos.ufrj.br (C.M.H. de Figueiredo), mazzuoccolo@unimore.it (G. Mazzuoccolo), Myriam.Preissmann@g-scop.inpg.fr (M. Preissmann), viniciussantos@decom.cefetmg.br (V.F. dos Santos), diana.sasakidesouzapereira@dauphine.fr (D. Sasaki).

http://dx.doi.org/10.1016/j.dam.2015.10.013
0166-218X/© 2015 Elsevier B.V. All rights reserved.

in [10] that for every subcubic graph G there exists an equitable k-total-coloring of G for each $k \geq \Delta(G) + 2$. This proves the validity of a conjecture of Fu [7] in the case of subcubic graphs. We show in Section 3 that it is NP-complete to decide whether the equitable total chromatic number of a bipartite cubic graph is equal to 4.

Since, by definition, the equitable total chromatic number of a graph cannot be smaller than its total chromatic number, we get that if a cubic graph has no 4-total-coloring, then it has a 5-total-coloring that is an equitable total-coloring. On the other hand, the equitable total chromatic number of Type 1 cubic graphs could be either 4 or 5. We devote Section 4 to this topic. Although there were known examples of graphs such that the total chromatic number is strictly less than the equitable total chromatic number [7], none of the examples found so far were cubic. Furthermore, it is known that the chromatic number is equal to the equitable chromatic number for all connected cubic graphs [4] and that the chromatic index is equal to the equitable chromatic index for any graph [17] (for more information about equitable colorings, we refer to [8]). So it was natural to consider the question of the existence of Type 1 cubic graphs of equitable total chromatic number 5. In Section 4.1, we present a construction that allows to obtain infinitely many such graphs.

For two infinite classes of Type 1 cubic graphs, ladder graphs [5] and Goldberg graphs [9], the known 4-total-colorings were not equitable. In Section 4.2 we show that all these graphs have equitable total chromatic number 4.

The results in Section 4 lead us to the following question.

Question 1. *Does there exist a Type* 1 *cubic graph with girth greater than* 4 *and equitable total chromatic number* 5*?*

2. Definitions

A *semi-graph* is a 3-tuple $G = (V(G), E(G), S(G))$ where $V(G)$ is the set of vertices of G, $E(G)$ is a set of edges having two distinct endpoints in $V(G)$, and $S(G)$ is a set of *semi-edges* having one endpoint in $V(G)$. When there is no chance of ambiguity, we simply write V, E or S.

We write edges having endpoints v and w shortly as vw. When vertex v is an endpoint of $e \in E \cup S$ we say that v and e are *incident*. Two elements of $E \cup S$ incident to the same vertex, respectively two vertices incident to the same edge, are called *adjacent*.

A *graph* G is a semi-graph with an empty set of semi-edges. In that case we can write $G = (V, E)$. Given a semi-graph $G = (V, E, S)$, we call the graph (V, E) the *underlying graph of* G.

All definitions given below for semi-graphs, that do not require the existence of semi-edges, are also valid for graphs. Semi-graphs will be used mostly in Section 4.

Let $G = (V, E, S)$ be a semi-graph. The *degree* $d(v)$ of a vertex v of G is the number of elements of $E \cup S$ that are incident to v. We say that G is *d-regular* if the degree of each vertex is equal to d. In this paper we are mainly interested in 3-regular graphs and semi-graphs, also called respectively *cubic graphs* and *cubic semi-graphs*. A graph whose vertices have degree at most 3 is called *subcubic graph*. Given a graph G of maximum degree 3, the semi-graph obtained from G by adding $3 - d(v)$ semi-edges with endpoint v, for each vertex v of G, is called *the cubic semi-graph generated by* G.

For $k \in \mathbb{N}$, a *k-vertex-coloring* of G is a map $C^V : V \to \{1, 2, \ldots, k\}$, such that $C^V(x) \neq C^V(y)$ whenever x and y are two adjacent vertices. The *chromatic number* of G, denoted by $\chi(G)$, is the least k for which G has a k-vertex-coloring.

Similarly, a *k-edge-coloring* of G is a map $C^E : E \cup S \to \{1, 2, \ldots, k\}$, such that $C^E(e) \neq C^E(f)$ whenever e and f are adjacent elements of $E \cup S$. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring. By Vizing's theorem [16] we have that $\chi'(G)$ is equal to either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices of G. If $\chi'(G) = \Delta(G)$, then G is said to be *Class* 1, otherwise G is said to be *Class* 2.

A *k-total-coloring* of G is a map $C^T : V \cup E \cup S \to \{1, 2, \ldots, k\}$, such that

(a) $C^T|_V$ is a vertex-coloring,
(b) $C^T|_{E \cup S}$ is an edge-coloring,
(c) $C^T(e) \neq C^T(v)$ whenever $e \in E \cup S$, $v \in V$, and e is incident to v.

The *total chromatic number* of G, denoted by $\chi''(G)$, is the least k for which G has a k-total-coloring. Clearly $\chi''(G) \geq \Delta(G) + 1$. The Total Coloring Conjecture [2] claims that $\chi''(G) \leq \Delta + 2$ and it has been proved for cubic graphs [12,15]. A cubic graph is said to be Type 1 or Type 2, according to the fact that its total chromatic number is 4 or 5, respectively.

A *proper partial k-coloring* of G is an assignment of at most k colors to some elements of G such that adjacent or incident elements have different colors.

A k-total-coloring is *equitable* if the cardinalities of any two color classes differ by at most one. The least k for which G has an equitable k-total-coloring is the *equitable total chromatic number* of G, denoted by $\chi_e''(G)$. We remind the reader of the conjecture that $\chi_e''(G) \leq \Delta + 2$ for any graph G, and that this conjecture was proved for cubic graphs [18], and recently for subcubic graphs [10].

The next proposition, which is about equitable $(k+1)$-total-colorings of k-regular graphs, will be used in Sections 3 and 4.

Proposition 1. *Let G be a k-regular graph of Type* 1 *and let C^T be a $(k + 1)$-total-coloring of G. The following three statements are equivalent.*

Download English Version:

https://daneshyari.com/en/article/418459

Download Persian Version:

https://daneshyari.com/article/418459

Daneshyari.com