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Graph products of the trivariate total domination polynomial and related polynomials

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ABSTRACT

A vertex subset $W \subseteq V$ of the graph $G = (V, E)$ is a total dominating set if every vertex of the graph is adjacent to at least one vertex in W. The total domination polynomial is the ordinary generating function for the number of total dominating sets in the graph. We investigated some graph products for a generalization of the total domination polynomial, called the trivariate total domination polynomial. We also show that the chromatic polynomial is encoded in the independent domination polynomial of some graph products. These results have a wide applicability to other domination related graph polynomials, e.g. the domination polynomial, the independent domination polynomial or the independence polynomial.

1. Introduction

The total domination number of graph products, especially for Cartesian products, has been studied in several papers (see [9,13,15]), but no attention has so far been given to the total domination polynomial of product graphs. In this paper we prove some theorems about graph products for a generalization of the total domination polynomial. This gives us information about the domination polynomial and the total domination polynomial of these graphs. Additionally, some results for the domination and the independent domination polynomial will be presented.

We consider finite simple undirected graphs and identify edges with two-element subsets of the vertex set. Let $G = (V, E)$ be a graph with vertex set V and edge set E. Furthermore, let $n(G) = |V|$ be the number of vertices and $m(G) = |E|$ be the number of edges of the graph G. If there is no risk of confusion we simply write n and m for the number of the vertices and the number of the edges, respectively. Let $W \subseteq V$, then the *(vertex) induced subgraph* $G[W]$ is the graph

$$G[W] = (W, \{e \in E : e \subseteq W\}).$$

We denote the number of isolated vertices in G by $\mathrm{iso}(G)$. The *open neighborhood* $N_G(v)$ of a vertex $v \in V$ is the set of all vertices that are adjacent to v in G. Analogously, we define

$$N_G(W) = \bigcup_{v \in W} N_G(v)$$

for any vertex subset $W \subseteq V$. If the graph is known from the context, we write $N(v)$ and $N(W)$ instead of $N_G(v)$ and $N_G(W)$, respectively. The maximum degree $\Delta(G)$ of the graph G is defined as $\max_{v \in V} |N(v)|$.

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A vertex set $W \subseteq V$ is called a *total dominating set* of G if $N_G(W) = V$. The *total domination polynomial* of a graph G is the ordinary generating function for the number of total dominating sets of G:

$$\mathrm{D}_t(G, x) = \sum_{\substack{W \subseteq V \\ N_G(W)=V}} x^{|W|}.$$

A generalization of the total domination polynomial offers a way to find a connection from the total domination polynomial to other graph polynomials. One possible and promising generalization is the trivariate total domination polynomial [5] (see [7] for another possible generalization).

Definition 1.1. Let $G = (V, E)$ be a graph. The trivariate total domination polynomial is given as follows:

$$\mathrm{Y}_t(G; x, y, z) = \sum_{W \subseteq V} x^{|W|} y^{|N(W)\setminus W|} z^{\mathrm{iso}(G[W])}.$$

We can get the total domination polynomial from the trivariate total domination polynomial by

$$\mathrm{D}_t(G, x) = [y^n]\mathrm{Y}_t(G; xy, y, 0). \tag{1}$$

Further polynomials, which can be extracted, are the domination polynomial (see [1])

$$D(G, x) = [y^n]\mathrm{Y}_t(G; xy, y, 1) \tag{2}$$

and the independent domination polynomial $\mathrm{D}_i(G, x)$, which is the ordinary generation function for the number of independent dominating subsets of a graph [22],

$$\mathrm{D}_i(G, x) = [y^n z^n]\mathrm{Y}_t(G; xy, yz, z). \tag{3}$$

An independent set of the graph $G = (V, E)$ is a vertex subset W, such that the vertices in W are pairwise non-adjacent. The independence polynomial [10,14] of G is defined by

$$\mathrm{I}(G, x) = \sum_{\substack{W \subseteq V \\ W \text{ ind. in } G}} x^{|W|}.$$

The trivariate total domination polynomial also has a nice connection to the independence polynomial (see [5]):

$$\mathrm{I}(G, x) = \lim_{z\to\infty} \mathrm{Y}_t\left(G; \frac{x}{z}, 1, z\right). \tag{4}$$

In this paper we investigate three different graph products: the Cartesian product, the lexicographic product and the strong product. Let G and H be graphs, then the vertex set of all of these product graphs is the Cartesian product of the vertex sets of G and H. In respect to the grid structure of the product graphs we call the vertex subset $\{(v, w) : w \in V(H)\}$, for a given vertex $v \in V(G)$, the vth row of the product graph $G \times H$. In the same manner we denote the set $\{(v, w) : V \in V(G)\}$, for a $w \in V(H)$, the wth column. We also assume that the vertices of the graphs are linearly ordered and therefore we call the set $\{(1, w) : w \in V(H)\}$ the first row of the product graph.

2. The Cartesian product

Our main result is a couple of equations for the calculation of the trivariate total domination polynomial of some graph products, e.g. the Cartesian and the lexicographic product. See [11] for a detailed overview about graph products and their applications. Due to the fact that the trivariate total domination polynomial is connected to other graph polynomials, these results have a wide applicability. Until recently, only few results for the Cartesian and the strong product with respect to the domination polynomial were known [17].

Definition 2.1. The Cartesian product $G \,\Box\, H$ of the graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is a graph such that

1. the vertex set of $G\Box H$ is $V(G) \times V(H)$ and
2. two vertices (u, v) and (x, y) are adjacent in $G\Box H$ if and only if either $ux \in E(G)$ and $v = y$ or $u = x$ and $vy \in E(H)$.

The most well known conjecture and main problem of the domination theory is the Vizing's conjecture [23].

Conjecture 2.2. *Let G and H be two graphs, then $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.*

There are a lot of papers with partial results about the conjecture, but no one has found a proof yet. In this paper we will prove one theorem for the product $K_2\Box K_n$ of the trivariate total domination polynomial and additionally two nice results of the independent domination polynomial.