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Poonen's conjecture and Ramsey numbers

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ABSTRACT

For $c \in \mathbb{Q}^*$, let $\varphi_c : \mathbb{Q} \to \mathbb{Q}$ denote the quadratic map $\varphi_c(X) = X^2 + c$. How large can the period of a rational periodic point of φ_c be? Poonen conjectured that it cannot exceed 3. Here, we tackle this conjecture by graph-theoretical means with the Ramsey numbers $R_k(3)$. We show that, for any $c \in \mathbb{Q}^*$ whose denominator admits at most k distinct prime factors, the map φ_c admits at most $2R_k(3) - 2$ periodic points. As an application, we prove that Poonen's conjecture holds for all $c \in \mathbb{Q}^*$ whose denominator is a power of 2.

1. Introduction

Let $c \in \mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. Our purpose in this paper is to tackle a problem in number theory, namely the study of rational periodic points under iteration of the quadratic map $\varphi_c : \mathbb{Q} \to \mathbb{Q}$, $\varphi_c(X) = X^2 + c$, by means of graph-theoretic arguments. The number of such periodic points is finite, as follows from a general result due to Northcott [4]. Here we obtain an upper bound on this number in terms of *a priori* unrelated bounds on graphs, namely the Ramsey numbers $R_k(3)$. Recall that $R_k(3)$ denotes the least positive integer N such that, for any k-coloring of the edges of the complete graph K_N on N vertices, there is a monochromatic triangle.

Here are a few more formal details. As usual, for $n \in \mathbb{N}$ we denote by $\varphi_c^n = \varphi_c \circ \cdots \circ \varphi_c$ the nth iterate of φ_c.

Definition 1.1. A *rational periodic point* of φ_c is an $x \in \mathbb{Q}$ such that there exists $n \in \mathbb{N}^*$ with $\varphi_c^n(x) = x$. The smallest such n is called the *period* of x. We shall denote by $\mathrm{Per}(\varphi_c)$ the set of all rational periodic points of φ_c, i.e.

$$\mathrm{Per}(\varphi_c) = \{x \in \mathbb{Q} \mid \exists n \in \mathbb{N}^* \text{ with } \varphi_c^n(x) = x\}.$$

Our main result in this paper is a bound on $|\mathrm{Per}(\varphi_c)|$ in terms of the Ramsey numbers $R_k(3)$, namely: given $c \in \mathbb{Q}^*$, let k be the number of distinct prime factors of the denominator of c. Then

$$|\mathrm{Per}(\varphi_c)| \le 2R_k(3) - 2.$$

As for individual rational periodic points in $\mathrm{Per}(\varphi_c)$, there is a very intriguing conjecture about their possible periods [1].

Conjecture 1.2 (*Poonen*)**.** *Let $c \in \mathbb{Q}^*$. Let $x \in \mathrm{Per}(\varphi_c)$. Then x is of period at most 3.*

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For an example of a periodic point of period 3, take $c = -29/16$ and $x_0 = -1/4$. Then the orbit of x_0 under iteration of φ_c is $\{-1/4, -7/4, 5/4\}$.

Here are a few partial results towards the conjecture. The relatively easy first one shows that the conjecture remains open only in the case where the denominator of c is even.

Proposition 1.3 ([7]). *If the denominator of c is odd, then all $x \in \mathrm{Per}(\varphi_c)$ have period at most 2. In particular, Poonen's conjecture holds in this case.*

Much more demanding are the results stating that no $x \in \mathrm{Per}(\varphi_c)$ may have period 4 (see [3]) or period 5 (see [1]). Moreover, period 6 would also be excluded, provided the Birch and Swinnerton-Dyer conjecture be shown to hold in one specific case [6].

As a corollary of our bound on $|\mathrm{Per}(\varphi_c)|$, we shall prove that Poonen's conjecture holds for all $c \in \mathbb{Q}^*$ whose denominator is a power of 2, and in fact of any prime p, even though the case p odd follows from Proposition 1.3.

2. On families of 2-stable graphs

Let G be a simple graph on the vertex set V. A subset $S \subseteq V$ is *stable* if no two vertices in S are neighbors in G. The largest cardinality of a stable set in G is called the *stability number* or *independence number* of G and is traditionally denoted $\alpha(G)$.

Definition 2.1. We say that a graph G is r-*stable* if $\alpha(G) \le r$, i.e. if G contains no stable set of size $r+1$.

For instance, a complete graph is 1-stable, and conversely. In this paper, we consider families of 2-stable graphs, and look for conditions forcing them to have a common edge. As usual, we shall denote by $E(G)$ the set of edges of a graph G.

Proposition 2.2. *Let $G_1, \dots, G_k$ be 2-stable simple graphs on a vertex set V. If $|V| \ge R_k(3)$, then $G_1, \dots, G_k$ have an edge in common.*

Proof. A simple yet key remark is that *a graph G on V is 2-stable if and only if its complement graph $\overline{G}$ on V is triangle-free*. Let now $N = |V|$, and assume that the G_i's have no common edge. Hence, for every edge $xy \in E(K_N)$, there exists an index $j \in \{1, 2, \dots, k\}$ such that $xy \notin E(G_j)$. Setting $\chi(xy) = j$ for any choice of such an index j defines a k-coloring

$$\chi : E(K_N) \to \{1, 2, \dots, k\}$$

of the edges of K_N, with the property that if $\chi(xy) = j$ then $xy \in E(\overline{G_j})$. Since the $\overline{G_j}$'s are triangle-free, it follows that $E(K_N)$ contains no monochromatic triangle under χ. Therefore, we must have $N < R_k(3)$, by the defining property of this Ramsey number. ■

We end this short section with some remarks on the numbers $R_k(3)$. Currently, their only exactly known values are $R_1(3) = 3$, $R_2(3) = 6$ and $R_3(3) = 17$. For $k = 4$, it is only known that $51 \le R_4(3) \le 62$. See [5]. More generally, the following bounds hold:

$$3.1999^k \le R_k(3) \le 2 - k + kR_{k-1}(3).$$

The lower and upper bound are taken from [9] and [2], respectively. See also [5,8] for more information.

3. An arithmetic consequence

We shall now use the Ramsey numbers $R_k(3)$ in a purely arithmetic statement. It implies, for instance, that there cannot exist three distinct odd positive integers x, y, z such that $x+y, x+z, y+z$ are all powers of 2. This follows from the next result for $P = \{2\}$ using the value $R_1(3) = 3$.

Notation 3.1. *For $n \in \mathbb{N}^*$, we denote by* $\mathrm{supp}(n)$ *the set of prime numbers p dividing n.*

For instance, $\mathrm{supp}(1) = \emptyset$, $\mathrm{supp}(64) = \{2\}$ and $\mathrm{supp}(120) = \{2, 3, 5\}$.

Theorem 3.2. *Let P be a finite set of prime numbers. Let $V \subseteq \mathbb{N}^*$ be a set of positive integers satisfying the following two conditions:*

(1) $\mathrm{supp}(x) \cap P = \emptyset$ *for all* $x \in V$,
(2) $\mathrm{supp}(x+y) \subseteq P$ *for all distinct* $x, y \in V$.

Then $|V| \le R_k(3) - 1$, where $k = |P|$.

Proof. We first define k simple graphs on V as vertex set. For each $p \in P$, let G_p be the graph on V such that, for all distinct $x, y \in V$,

$$\{x, y\} \in E(G_p) \iff \begin{cases} x + y \not\equiv 0 \bmod p & \text{if } p \text{ odd}, \\ x + y \not\equiv 0 \bmod 4 & \text{if } p = 2. \end{cases}$$