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Locating sets of vertices on Hamiltonian cycles

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ABSTRACT

Given a fixed positive integer $k \geq 2$ and a fixed pair of sets of vertices $X = \{x_1, x_2, \cdots, x_k\}$ and $Y = \{y_1, y_2, \cdots, y_k\}$ in a graph G of sufficiently large order n, the sharp minimum degree condition $\delta(G) \geq (n+k-1)/2$ will be shown to imply the existence of a Hamiltonian cycle C such that all of the vertices of X precede the vertices of Y for appropriate initial vertex and orientation of the cycle C. Also, a minimum degree condition along with a connectivity condition will be shown to imply the existence of a Hamiltonian cycle C such that the vertices of X and Y alternate on the cycle C.

1. Introduction

We deal only with finite simple graphs and our notation generally follows the notation of Chartrand and Lesniak in [1]. Given an ordered set of vertices $S = \{x_1, x_2, \ldots, x_k\}$ in a graph, there are a series of results giving minimum degree conditions that imply the existence of a Hamiltonian cycle such that the vertices in S are located in order on the cycle with restrictions on the distance between consecutive vertices of S. Examples include results by Kaneko and Yoshimoto [5], Sárközy and Selkow [7], Kierstead, Sárközy and Selkow [6], Faudree, Gould, Jacobson and Magnant [2], Faudree and Li [4], and Faudree, Lehel, and Yoshimoto [3]. We will consider a pair of disjoint sets of vertices X and Y, each with precisely $k \geq 2$ vertices in a graph G of order n. The objective is to determine the minimum degree $\delta(G)$ of G that implies the existence of a Hamiltonian cycle C such that the smallest interval of C that contains X and the smallest interval of C that contains Y are disjoint.

The following result will be proved in Section 2.

Theorem 1. *Let k be a positive integer. If G is a graph with $n \geq 5k+2$ and $\delta(G) \geq (n+k-1)/2$, then for any two disjoint sets X and Y of k vertices each in G, there exists a Hamiltonian cycle C of G, such that the vertices of X precede the vertices of Y for appropriate initial vertex and orientation of the cycle C.*

A companion to the results of Theorem 1 is to place the vertices of X and Y on a cycle that the vertices alternate between being in X and being in Y, called *an alternating cycle for X and Y*. This is related to the concept of being p-ordered.

Definition 1. A graph G is p-ordered Hamiltonian if for any ordered set of p vertices in G, there is a Hamiltonian cycle such that the p vertices are encountered in the specified order.

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The following result due to Kierstead, Sárközy and Selkow [6, Theorem 1] gives a minimum degree condition that implies a graph is p-ordered.

Theorem A. *If $p \geq 2$ and $n \geq 11p - 3$, and G is a graph of order n with $\delta(G) \geq \lceil n/2 \rceil + \lfloor p/2 \rfloor - 1$, then G is p-ordered Hamiltonian.*

This immediately give the following corollary.

Corollary 1. *Let $k \geq 2$ and let $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ be two sets of k disjoint vertices in a graph G of order n. If $\delta(G) \geq (n + 2k - 2)/2$, and $n \geq 22k - 3$, then there exists an alternating Hamiltonian cycle for X and Y.*

In the case of alternating the vertices between X and Y on a Hamiltonian cycle, the minimum degree can be lowered in the presence of sufficient connectivity. Suppose a graph G has a cut set S of at most $2k - 1$ vertices such that $G - S$ has two components G_1 and G_2 with $\min\{|G_1|, |G_2|\} \geq k$, where $|G_i|$ is the order of G_i. If we choose X and Y of k vertices from G_1 and G_2, respectively, then obviously there is no alternating cycle for X and Y. Thus we need connectivity at least $2k$ for the existence of such a cycle.

Thomas and Wollan [8, Corollary 1.2] showed the following:

Theorem B. *If G is a $2k$-connected graph with $|E(G)| \geq 5kn$, then G is k-linked, i.e., for any two sets $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ of k disjoint vertices in G, there exist disjoint k paths $P_1, \ldots, P_k$ such that P_i joins x_i and y_i for all $1 \leq i \leq k$.*

This implies that a $4k$-connected graph G with $\delta(G) \geq 20k$ is $2k$-linked. Let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ be two sets of k disjoint vertices in a graph G. If $\delta(G) \geq 4k - 1$, then there is a matching $M = \{x_i x_i', y_i y_i' : 1 \leq i \leq k\}$ covering $X \cup Y$ in G. If G is $2k$-linked, then there exist internally disjoint $2k$ paths $P_1, \ldots, P_k, Q_1, \ldots, Q_k$ such that P_i joins x_i and y_i and Q_i joins y_i' and x_{i+1}' for $1 \leq i \leq k$. The union of those $2k$ paths and M constructs an alternating cycle for X and Y. Thus the following lemma holds.

Lemma 1. *If G is a $4k$-connected graph with $\delta(G) \geq 20k$, then for any disjoint sets X and Y of k vertices of G, there exists an alternating cycle for X and Y.*

In Section 3, we will prove the following lemma.

Lemma 2. *Let G be a $2k$-connected graph with $\delta(G) \geq (n + 1)/2$ and $n \geq 10k - 1$, and let X and Y be disjoint sets of k vertices of G. If G has an alternating cycle for X and Y, then G has an alternating Hamiltonian cycle for X and Y.*

Lemmas 1 and 2 imply the second result of this paper.

Theorem 2. *If G is a $4k$-connected graph with $\delta(G) \geq \max\{(n + 1)/2, 20k\}$ and $n \geq 10k - 1$, then for any disjoint sets X and Y of k vertices of G, there exists an alternating Hamiltonian cycle for X and Y.*

It is easy to see the above result implies the following.

Corollary 2. *If G is a $4k$-connected graph with $\delta(G) \geq (n + 1)/2$ and $n \geq 40k - 1$, then for any disjoint sets X and Y of k vertices of G, there exists an alternating Hamiltonian cycle for X and Y.*

The following examples show that the minimum degree conditions are sharp in Theorem 1 and Corollary 1.

Example 1. For $k \geq 2$ consider the graph $G_1 = K_k + (K_{(n-k)/2} \cup K_{(n-k)/2})$. Assume that the vertices of X are in the K_k and the vertices of Y are split between the two complete graphs $K_{(n-k)/2}$. Then, there is no Hamiltonian cycle C of G_1 for which the vertices of X precede the vertices of Y for appropriate initial vertex and orientation of the cycle C. Thus, the minimum degree condition of Theorem 1 cannot be lowered.

Example 2. For $k \geq 2$ consider the graph $G_2 = K_{2k-1} + (K_{(n-2k+1)/2} \cup K_{(n-2k+1)/2})$. Assume that the vertices of X are in one of the copies of $K_{(n-2k+1)/2}$ of G and the vertices of Y are in the other copy of $K_{(n-2k+1)/2}$. Then, there is no Hamiltonian cycle C of G for which the vertices of X and Y alternate on the cycle C, since this would require that the cut set K_{2k-1} have $2k$ vertices. Thus, the minimum degree condition of Corollary 1 cannot be lowered.

For Theorem 2, we raise two natural questions.

Question 1. *What are the sharp minimum degree conditions and connectivity conditions in Theorem 2 that imply the existence of an alternating Hamiltonian cycle for X and Y in a graph G.*

Question 2. *Clearly $\delta(G) \geq n/2$ is sufficient for there to be a Hamiltonian cycle, and $\kappa(G) \geq 2k$ by Example 2 for the vertices of X and Y to alternate on the Hamiltonian cycle. Are these conditions sufficient for the existence of an alternating Hamiltonian cycle for X and Y?*