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On bipartization of cubic graphs by removal of an independent set[☆]

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ABSTRACT

We study a new problem for cubic graphs: bipartization of a cubic graph Q by deleting sufficiently large independent set I. It can be expressed as follows: *Given an integer k and a connected n-vertex tripartite cubic graph $Q = (V, E)$ with independence number $\alpha(Q)$, does Q contain an independent set I of size k such that $Q - I$ is bipartite?* We are interested for which values of k the answer to this question is affirmative. We prove constructively that if $\alpha(Q) \geq 2n/5$, then the answer is positive for each k satisfying $\lfloor (n-\alpha(Q))/2 \rfloor \leq k \leq \alpha(Q)$. It remains an open question if a similar construction is possible for $\alpha(Q) < 2n/5$.

We also show that this problem with $\alpha(Q) \geq 2n/5$ and k satisfying $\lfloor n/3 \rfloor \leq k \leq \alpha(Q)$ can be related to semi-equitable graph 3-coloring, where one color class is of size k, and the subgraph induced by the remaining vertices is equitably 2-colored. This means that Q has a coloring of type $(k, \lceil (n-k)/2 \rceil, \lfloor (n-k)/2 \rfloor)$.

1. Some preliminaries

There are many challenging and interesting problems involving independent sets and cubic graphs. One of the most known is the problem of independence, $\mathrm{IS}(Q, k)$:

Given a connected cubic graph $Q = (V, E)$ and an integer k, does Q contain an independent set of size at least k?

An *independent set* of a graph Q is a subset I of the vertices of Q, $I \subseteq V(Q)$, such that no two vertices in I are joined by an edge in Q. The size of the largest independent set is called the *independence number* of Q, and it is denoted by $\alpha(Q)$. The problem of finding the value of $\alpha(Q)$ is widely discussed in the literature. In general, the problem $\mathrm{IS}(Q, k)$ is NP-complete for cubic graphs, and even for planar cubic graphs [5]. A comprehensive survey of results on the IS problem, including cubic graphs, was presented in [1,8,10].

The second type of problems is connected with decycling sets of cubic graphs (also known as feedback-vertex sets). For a graph Q, a subset $S \subseteq V(Q)$ is a *decycling set* of Q if and only if $Q - S$ is acyclic, where by $Q - S$ we mean the subgraph of Q induced by the vertices in $\overline{S} = V(Q) \setminus S$. Although the decycling set decision problem is NP-complete in general, it is polynomially solvable for cubic graphs [11].

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The third group contains problems connected with *bipartization* of cubic graphs. Given a graph, the task is to find a smallest set of vertices whose deletion makes the remaining graph bipartite. Choi et al. [4] showed that the bipartization decision problem is NP-complete for cubic graphs. Some approximation algorithms were given in [9].

In this paper we combine the above approaches and define the Bipartization IS problem BIS(Q, k), as follows:

> Given a connected cubic graph $Q = (V, E)$ and integer k, does Q contain an independent set I of size at least k such that $Q - I$ is bipartite?

We are interested for which values of k the answer to this question is affirmative. This problem can be seen as a task of finding independent odd decycling sets.

We say that a graph G is *t-colorable* if its vertex set can be partitioned into t independent sets-color classes. The smallest value of t admitting t-colorability of graph G is named the *chromatic number* of G and denoted by $\chi(G)$. Let us recall Brooks' theorem:

Theorem 1 *([2]). For any connected graph G with maximum degree Δ, the chromatic number $\chi(G)$ of G is at most Δ, unless G is a clique or an odd cycle.*

This implies that

$$2 \leq \chi(Q) \leq 3$$

for all cubic graphs except K_4.

It is obvious that for 2-chromatic cubic graphs and $k \leq |V(Q)|/2$ the answer to BIS(Q, k) is affirmative. Hence, in the sequel we consider only connected cubic graphs Q with $\chi(Q) = 3$. This means that $V(Q)$ can be partitioned into three independent sets and Q is not bipartite. The class of such cubic graphs will be denoted by $\mathcal{Q}_3$. Its subclass of graphs on n vertices will be denoted by $\mathcal{Q}_3(n)$.

A graph is *equitably t-colorable* if and only if its vertex set can be partitioned into independent sets $V_1, V_2, \ldots, V_t$ such that $\big||V_i| - |V_j|\big| \leq 1$ for all $i, j = 1, 2, \ldots, t$. The smallest value of t admitting such coloring of the graph G is named *equitable chromatic number* of G and denoted by $\chi_=(G)$.

In the case of cubic graphs we know that

$$\chi_=(Q) = \chi(Q), \tag{1}$$

where $\chi_=(Q)$ is the equitable chromatic number of Q. This follows from

Theorem 2 *(Chen, Lih, and Wu, 1994, [3]). Every proper coloring of connected cubic graph can be made equitable without adding new colors.*

Chen et al.'s [3] algorithm relies on repeatedly decreasing the width of coloring (the difference between the cardinality of the largest and smallest color class) by one until the difference is not greater than one.

In this paper we are also interested in equitable coloring of $Q - I$. We will present an algorithm which, given an independent set of size $k \geq 2n/5$, constructs an appropriate independent set I of size k for the BIS(Q, k) problem with $Q \in \mathcal{Q}_3(n)$. We will also prove that such cubic graphs have colorings of type $(k, \lceil (n-k)/2 \rceil, \lfloor (n-k)/2 \rfloor)$, which means that $Q - I$ has an equitable 2-coloring. Such type of coloring is called *semi-equitable*, i.e. the coloring in which exactly one color class is of any size while the cardinalities of the remaining color classes differ by at most 1. Colorings of this kind are useful in a problem of scheduling identical jobs on three parallel uniform processors [7]. In such a model of scheduling one of processors is faster than the remaining two, while the two slower processors are of the same speed and the conflict graph is cubic.

2. Main results

Our main result concerning BIS(Q, k) is as follows.

Theorem 3. *If $Q \in \mathcal{Q}_3(n)$ and $\alpha(Q) \geq 2n/5$, then there exists an independent set I of size k in Q such that $Q - I$ is bipartite for $\lfloor n/3 \rfloor \leq k \leq \alpha(Q)$.*

Note, that this leaves the problem open for $\lceil n/3 \rceil \leq \alpha(Q) < 2n/5$.

Before we prove Theorem 3, we need some auxiliary concepts.

We consider connected cubic graphs $Q \in \mathcal{Q}_3(n)$ with independence number $\alpha(Q) \geq 2n/5$, and let I be an independent set of size at least $2n/5$. If $Q - I$ is not bipartite, then the subgraph $Q - I$ consists of two parts: a 2-chromatic part of all bipartite components and a 3-chromatic part containing odd cycles (possibly with chords, bridges, pendant edges, etc.).

Definition 1. For $Q \in \mathcal{Q}_3$, the *residuum $R(I)$ of Q with respect to an independent set I* is the set of all odd cycles in the graph $Q - I$.

For example, for the graph in Fig. 1 and given I, $R(I) = \{v_1v_2v_3, v_4v_5v_6\}$.