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A survey on flows in graphs and matroids

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ABSTRACT

A quintessential minimax relation is the Max-Flow Min-Cut Theorem which states that the largest amount of flow that can be sent between a pair of vertices in a graph is equal to the capacity of the smallest bottleneck separating these vertices. We survey generalizations of these results to multi-commodity flows and to flows in binary matroids. Two tantalizing conjectures by Seymour on the existence of fractional and integer flows are motivating our work. We will not assume from the reader any background in matroid theory.

1. Flows in graphs

In a flow problem we are given a graph G where the edges are partitioned into *demand* edges Σ and *capacity* edges $E(G) - \Sigma$.[1] Every edge e is assigned a non-negative integer value w_e. For a demand edge e, w_e indicates the amount of flow required between the endpoints of e, and for a capacity edge e, w_e is the maximum amount of flow allowed on that edge. The triple (G, Σ, w) is a *flow instance*. An example of a flow instance is given in Fig. 1. For every circuit C that contains exactly one demand edge $e = st$ we assign some non-negative value y_C that indicates the amount of flow that is carried between s and t along the path $C - \{e\}$. (By a circuit we mean a set of edges that forms a connected graph where every vertex has degree two.) Denote by $\mathscr{C}$ the set of all circuits that contain exactly one demand edge. Then $y \in \Re^{\mathscr{C}}$ is a *flow* if $y \geq \mathbf{0}$ and the following constraints hold:

Capacity constraints: for every edge $e \in E(G) - \Sigma$,

$$\sum (y_C : e \in C \in \mathscr{C}) \leq w_e; \tag{1}$$

Demand constraints: for every edge $e \in \Sigma$,

$$\sum (y_C : e \in C \in \mathscr{C}) = w_e. \tag{2}$$

Here (1) ensures that the total flow carried by a capacity edge e does not exceed its capacity w_e, and (2) guarantees that the total flow carried between the endpoints of a demand edge e is equal to its demand w_e. Consider the flow instance in Fig. 1. Set

$$\begin{aligned} y_{C_1} &= 1 \quad \text{for } C_1 = \{ac, cb, ba\} \\ y_{C_2} &= 1 \quad \text{for } C_2 = \{ac, cd, dg, gf, fa\} \\ y_{C_3} &= 2 \quad \text{for } C_3 = \{fd, dg, gf\} \end{aligned}$$

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[1] Given sets A, B we denote by $A - B$ the set $\{a \in A : a \notin B\}$.

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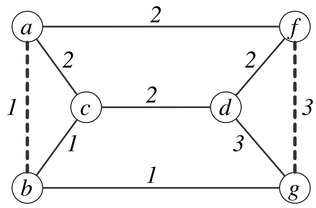

Fig. 1. Flow instance: demand edges Σ are dashed, w_e is indicated on edge e.

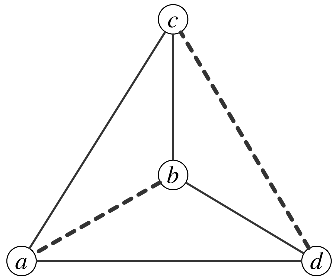

Fig. 2. Flow instance: demand edges are dashed, for every edge e, $w_e = 1$.

and $y_C = 0$ for all other circuits $C \in \mathscr{C}$. It can be readily checked that the capacity and the demand constraints are satisfied for this instance. Thus y is a flow.

Does every flow instance admit a flow? No. Indeed it can be readily checked that a necessary condition for the existence of a flow is that for every cut $\delta(U)$ the total demand across the cut should not exceed the total capacity across the cut, in other words,

$$w\big(\delta(U) \cap \Sigma\big) \le w\big(\delta(U) - \Sigma\big).^{2} \tag{3}$$

This is known as the *cut-condition*.

1.1. Integer flows

We say that flow y is *integer* if y is a vector with all integer entries. By a *fractional flow* we mean a flow y where y can take either integer or fractional values. Thus every integer flow is fractional but not vice-versa. A natural question is whether satisfying the cut-condition is sufficient to ensure the existence of an integer flow. The example in Fig. 2 shows that this is not the case. Here all the demand and capacity edges e are assigned the value $w_e = 1$. Note, that every cut contains at least as many solid edges as dashed edges, hence the cut-condition holds. Suppose, for a contradiction there exists an integer flow y. By symmetry we may assume that $y_C = 1$ for $C = \{ab, bc, ca\}$. But then the capacity of both edges incident to c are fully used, and we cannot satisfy the demand for edge cd. Hence, there does not exist an integer flow.

Theorem 1 states that if a flow instance does not "contain" that bad instance then the cut-condition is sufficient for the existence of an integer flow. Before we can formally state that result we need to define a containment operation. A *signed graph* is a pair (G, Σ) where G is a graph and Σ is a subset of the edges. A set of edges B is *odd* if $|B \cap \Sigma|$ is odd and it is *even* otherwise. In particular, we will say that edges in Σ are *odd*. We define the following operations on a signed graph (G, Σ)[3]:

- *Deleting an edge e*: replacing (G, Σ) by $(G \backslash e, \Sigma - \{e\})$;
- *Contracting an edge $e \notin \Sigma$*: replacing (G, Σ) by $(G/e, \Sigma)$;
- *Resigning*: replacing (G, Σ) by $(G, \Sigma \triangle \delta(U))$ where $\delta(U)$ is a cut of G.[4]

A signed graph (H, Γ) is a *minor* of (G, Σ) if it can be obtained from (G, Σ) by repeatedly applying the aforementioned operations. An *odd-K_4* is the signed graph obtained from K_4 where all edges are odd, i.e. it is $(K_4, E(K_4))$. Given a flow instance (G, Σ, w) we view (G, Σ) as a signed graph where the demand edges are the odd edges. Observe that if we resign on the cut $\delta(\{a, b\})$ the bad instance in Fig. 2, we obtain odd-K_4. In particular, odd-K_4 is a minor of the bad instance.

We are now ready to state sufficient conditions for the existence of an integer flow (this a special case of Theorem 12 that will be described later).

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2 For any $S \subseteq E(G)$, $w(S) := \sum_{e \in S} w_e$.

3 Given a graph H, then H/e (resp. $H \backslash e$) is the graph obtained by contracting (resp. deleting) edge e.

4 For any sets A, B, $A \triangle B$ denotes the set $A \cup B - (A \cap B)$.