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Proof of Berge's path partition conjecture for $k \geq \lambda - 3$

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ABSTRACT

Let D be a digraph. A *path partition* of D is called k-optimal if the sum of the k-norms of its paths is minimal. The *k-norm* of a path P is $\min(|V(P)|, k)$. Berge's path partition conjecture claims that for every k-optimal path partition $\mathcal{P}$ there are k disjoint stable sets orthogonal to $\mathcal{P}$. For general digraphs the conjecture has been proven for $k = 1, 2, \lambda - 1, \lambda$, where λ is the length of a longest path in the digraph. In this paper we prove the conjecture for $\lambda - 2$ and $\lambda - 3$.

1. Introduction

Let $D = (V, A)$ be a digraph. A path partition of D is a set of disjoint (directed) paths $P_1, P_2, \ldots, P_m$ for which $V(P_1) \cup V(P_2) \cup \cdots \cup V(P_m) = V$. Throughout the paper by path we always mean directed path and a single vertex is also considered to be a path. Let $\mathcal{P}$ be a path partition and $\mathcal{S}$ be a set of k disjoint stable sets. We say that $\mathcal{P}$ and $\mathcal{S}$ are *orthogonal* if each path P_i intersects as many of the k stable sets as possible, i.e. $\min(|V(P_i)|, k)$. The Greene–Kleitman theorem [5] has shown that if the digraph is acyclic and transitive (i.e. represents a partially ordered set), then for each positive integer k and for each path partition $\mathcal{P}$ minimizing $\sum \min(|V(P_i)|, k)$ there are k disjoint stable sets orthogonal to $\mathcal{P}$. In 1982, Berge made his conjecture claiming the same for all digraphs [1].

The conjecture is known to be true for acyclic digraphs [6] and for $k \geq \lambda - \sqrt{\lambda}$ (where λ is the cardinality of a longest path in D) for strongly connected digraphs [9]. However, for general digraphs only four cases are known: $k = 1, 2, \lambda - 1, \lambda$ [4,2,9,7,3].

In this paper we introduce a new variation of the stability number and prove a min–max theorem which directly generalizes the Greene–Kleitman theorem for general directed graphs. We then use this result to prove the path partition conjecture for $k \geq \lambda - 3$.

We use the following definitions and notations:

Definition 1. The *k-norm* of a path partition $\mathcal{P} = \{P_1, \ldots, P_m\}$ is defined by:

$$|\mathcal{P}|_k = \sum \min(|V(P_i)|, k).$$

A path partition is *k-optimal* if its k-norm is minimal.

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Definition 2. For a digraph D, $\pi_k(D)$ denotes $|\mathcal{P}|_k$ where $\mathcal{P}$ is a k-optimal path partition of D.

Definition 3. Let $\mathcal{P}$ be a path partition and $S^1, \dots, S^k$ disjoint stable sets. We say that $S^1, \dots, S^k$ are *orthogonal* to $\mathcal{P}$ if each path P of $\mathcal{P}$ intersects exactly $\min(|V(P)|, k)$ sets of $S^1, \dots, S^k$.

Remark 1. Let $\mathcal{P}$ be a path partition and $S^1, \dots, S^k$ k disjoint stable sets orthogonal to $\mathcal{P}$. Then we have $\sum |S^i| \geq |\mathcal{P}|_k$. Indeed,

$$\sum_i |S^i| = \sum_{P\in\mathcal{P}} \sum_i |V(P) \cap S^i| \geq \sum_{P\in\mathcal{P}} \min(|V(P)|, k) = |\mathcal{P}|_k.$$

Definition 4. Let $\mathcal{P}$ be a path partition. We denote by $\mathcal{P}^{\leq k}$ the set of paths in $\mathcal{P}$ with cardinality at most k. Similarly we denote by $\mathcal{P}^{\geq k}$ the set of paths in $\mathcal{P}$ with cardinality at least k.

Conjecture 1 (*Berge's path Partition Conjecture*)**.** *Let D be a digraph and k a positive integer. Then for every k-optimal path partition $\mathcal{P}$ there are k disjoint stable sets orthogonal to $\mathcal{P}$.*

Finding a k-optimal path partition in general digraphs is NP-complete as $\pi_k(D) = k$ for any $k < n$ if and only if there is a Hamiltonian path in D. However, if we also allow cycles in our partition and thus consider path–cycle partitions, then finding a k-optimal path–cycle partition and k disjoint stable sets orthogonal to its paths can be done in polynomial time. In [3] E. Berger and I.B-A. Hartman gave a common proof for the $k = 1, \lambda - 1, \lambda$ cases by searching k-optimal path–cycle partitions in subdigraphs where a k-optimal path–cycle partition will contain no cycle and thus will be a path partition.

Our approach is similar in that aspect but follows a different path. We prove a min–max theorem between the k-optimal path–cycle partitions and a variation of stability number. Then we use it on a suitable maximal acyclic subdigraph to prove that for a k-optimal path partition with each of its paths either not longer than $k + 1$ vertices or not shorter than $\lambda - 1$ vertices, there are k disjoint stable sets orthogonal to it. As a special case of this result we will get that if $k \geq \lambda - 3$, then for any path partition we can either find k disjoint stable sets orthogonal to it or find a better path partition.

2. A min–max theorem

As stated above finding a k-optimal path partition is NP-hard. So instead we use a similar but easier to handle structure: the k-optimal path–cycle partitions.

On the other hand in Remark 1 we have seen that if $S^1, S^2, \dots, S^k$ are k disjoint stable sets orthogonal to a path partition $\mathcal{P}$ then $\sum |S^i| \geq |\mathcal{P}_k|$. Finding k disjoint stable sets such that the sum of their cardinality is at least m (where m is an input) is also NP-hard. So again we choose to replace it with a notion similar enough to be of use for us but easier to handle.

The advantage of this approach is that we can prove a min–max theorem for this easier to handle structures. In the next section we use this min–max theorem to prove our result for the original hard to handle structures.

Definition 5. The k-norm of a path–cycle partition $\mathcal{P}^c = \{P_1, \dots, P_r, C_1, \dots, C_t\}$ where P_i are paths and C_i are cycles is the following:

$$|\mathcal{P}^c|_k = \sum_{i=1}^{r} \min(|V(P_i)|, k).$$

A path–cycle partition is k-optimal if its k-norm is minimal.

Definition 6. Let D be a digraph. Then $\pi_k^c(D) = |\mathcal{P}^c|_k$ where $\mathcal{P}^c$ is a k-optimal path–cycle partition of D.

Let S be a stable set in the digraph $D = (V, A)$. We call a vertex set $S_{cut} \subseteq V$ an S-cut set (or just cut-set if S is unambiguous) if $S \cap S_{cut} = \emptyset$ and all directed paths from S to S contain a vertex from S_{cut} as an internal point — we say S_{cut} *cuts* every S to S dipath. The pair (S, S_{cut}) is called *stable-cut pair*. For a pair of stable set S and one of its cut-sets S_{cut}, we use the notation $\langle S, S_{cut}\rangle = |S| - |S_{cut}|$.

Definition 7. An (S, S_{cut}) stable-cut pair is *optimal* if $\langle S, S_{cut}\rangle$ is the largest possible.

Definition 8. Two stable-cut pairs (S^1, S^1_{cut}) and (S^2, S^2_{cut}) are *disjoint* if S^1 and S^2 are disjoint.