Discrete Applied Mathematics 209 (2016) 144–152

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Colour degree matrices of graphs with at most one cycle

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ARTICLE INFO

Article history:
Received 3 November 2014
Received in revised form 15 July 2015
Accepted 24 July 2015
Available online 17 August 2015

Keywords:
Colour degree matrix
Degree constrained edge-partitioning
Edge packing
Discrete tomography
Edge-disjoint realisations

ABSTRACT

Colour degree matrix problems, also known as edge-disjoint realisation and edge packing problems, have connections for example to discrete tomography. Necessary and sufficient conditions are known for a demand matrix to be the colour degree matrix of an edge-coloured forest. We give necessary and sufficient conditions for a demand matrix to be realisable by a graph with at most one cycle.

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1. Introduction

A *demand sequence* $(d_1, d_2, \ldots, d_n)$ of non-negative integers is called *graphical* if there is a (simple) graph G such that vertex v has degree d_v for each $v \in [n]$, where $[n] = \{1, \ldots, n\}$. Necessary and sufficient conditions are known for a sequence to be graphical [15,13,9,21], with corresponding polynomial time algorithms to test these conditions and to find a realisation if there is one [20,10].

We consider an extension involving edges coloured with c colours. Let c and n be positive integers. An $n \times c$ *demand matrix* is a matrix $D = (d_{v,q} : v \in [n], q \in [c])$ of non-negative integers. We call D a *colour degree matrix* if there exists a c-edge coloured graph G on $[n]$, which *realises* D; that is, for each $q \in [c]$, each vertex $v \in [n]$ has $d_{v,q}$ incident edges coloured q. Note that this colouring will not be proper if some $d_{v,q} \geq 2$. Finding a realisation of a demand matrix is also known as finding *edge-disjoint realisations* [12], *edge packing* [3] or *degree constrained edge-partitioning* [1]. This problem is closely connected to *discrete tomography* [7] with many applications in industry, ranging from image reconstruction to material science [16].

Recently it was shown that for any fixed $c \geq 2$, the problem of deciding whether a demand matrix is a colour degree matrix is NP-complete [8,5,11]. However, we may be interested in realisations where the graph has some special structure, and restricting the solution class may yield a tractable problem. Previous work has often focused on colour degree matrices with c fixed at two or three [12,1,6], but we allow general c. First we briefly discuss the already studied case of forests, after which we investigate the case of graphs with at most one cycle. This is of course still rather a restricted class of graphs, for example for applications in tomography, but at least it is a step beyond forests, and we find a richer structure.

1.1. Colour degree matrices of forests

The following result was observed by Harary and Menon [19,14], for the monochromatic case.

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http://dx.doi.org/10.1016/j.dam.2015.07.025
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Proposition 1.1. *A non-zero sequence* $(d_1, d_2, \ldots, d_n)$ *of non-negative integers is the degree sequence of a forest if and only if the total demand* $\sum_v d_v$ *is even and at most* $2s - 2$*, where s is the number of* v *such that* $d_v \neq 0$.

The conditions are clearly necessary, since a forest on $s \geq 1$ vertices has at most $s - 1$ edges. They are easily seen to be sufficient using an inductive argument, since if a non-zero sequence satisfies the conditions then there is a vertex v with $d_v = 1$.

When is a demand matrix the colour degree matrix of a forest? There are simple necessary conditions. Let c and n be positive integers, and consider an $n \times c$ demand matrix $D = (d_{v,q} : v \in [n], q \in [c])$. For each $I \subseteq [c]$, denote the demand of v for the set of colours I by $d_{v,I} = \sum_{q \in I} d_{v,q}$. Observe that $(d_{1,I}, \ldots, d_{n,I})$ is the demand sequence we obtain when we ignore colours not in I and identify colours in I. Clearly it is necessary that each such sequence $(d_{1,I}, \ldots, d_{n,I})$ is the degree sequence of a forest. These conditions were shown to be sufficient by Bentz et al. in [1] for the case $c = 2$, and by Carroll and Isaak in [4] for general c.

Theorem 1.2 ([1,4])**.** *The demand matrix* $D = (d_{v,q} : v \in [n], q \in [c])$ *is the colour degree matrix of a forest if and only if, for each* $I \subseteq [c]$*, the sequence* $(d_{1,I}, \ldots, d_{n,I})$ *is the degree sequence of a forest.*

We may restate this theorem in an apparently more quantitative way. For each $I \subseteq [c]$, let

$$S(I) = S_D(I) = \{v : d_{v,I} > 0\} = \{v : d_{v,q} > 0 \text{ for some } q \in I\}$$

be the *support* of I, and $s(I) = s_D(I) = |S_D(I)|$; and let

$$t(I) = t_D(I) = \sum_{v \in [n]} d_{v,I} = \sum_{q \in I} \sum_{v \in [n]} d_{v,q}.$$

For a single element $q \in [c]$ we will write simply $S(q)$, $s(q)$ and $t(q)$.

Theorem 1.3 ([1,4])**.** *The demand matrix* $D = (d_{v,q} : v \in [n], q \in [c])$ *is the colour degree matrix of a forest if and only if*

1. $t(q)$ *is even for each* $q \in [c]$*, and*
2. $t(I) \leq 2s(I) - 2$ *for each* $I \subseteq [c]$ *with* $t(I) > 0$.

1.2. Colour degree matrices of graphs with at most one cycle

The following result was shown by Harary and Boesch [2], for the monochromatic case.

Proposition 1.4. *A sequence* $(d_1, d_2, \ldots, d_n)$ *of non-negative integers is the degree sequence of a graph with at most one cycle if and only if*

(a) *the total demand* $\sum_v d_v$ *is even and at most* $2s$*, where s is the number of* v *such that* $d_v \neq 0$*, and*
(b) *if the total demand is* $2s > 0$ *then there are at least* 3 *vertices with demand at least* 2.

The conditions are clearly necessary, since a graph on s vertices with at most one cycle has at most s edges. They are easily seen to be sufficient by an inductive argument as for forests, since if a non-zero sequence satisfies the conditions then either each demand is 2, or there is a vertex v with $d_v = 1$.

When is a demand matrix D the colour degree matrix of a graph with at most one cycle? This is the question on which we focus in this paper. As with forests, there are simple necessary conditions.

Clearly, for each $I \subseteq [c]$, it is necessary that $(d_{1,I}, \ldots, d_{n,I})$ is the degree sequence of a graph with at most one cycle. Further, call a set $I \subseteq [c]$ *critical* if $t(I) = 2s(I) > 0$. If I is critical then in any realisation G there must be a cycle where each edge has a colour in I. Thus another necessary condition is that there are no two disjoint critical (colour) sets. We shall see that these conditions are also sufficient.

Theorem 1.5. *The demand matrix* $D = (d_{v,q} : v \in [n], q \in [c])$ *is the colour degree matrix of a graph with at most one cycle if and only if*

(a) *for each* $I \subseteq [c]$ *the sequence* $(d_{1,I}, \ldots, d_{n,I})$ *is the degree sequence of a graph with at most one cycle, and*
(b) *there are no two disjoint critical sets.*

Let us restate Theorem 1.5, much as we restated Theorem 1.2 for forests. Theorems 1.5 and 1.6 together form our main result.

Theorem 1.6. *The demand matrix* $D = (d_{v,q} : v \in [n], q \in [c])$ *is the colour degree matrix of a graph with at most one cycle if and only if*

(U1) $t(q)$ *is even for each* $q \in [c]$*,*
(U2) $t(I) \leq 2s(I)$ *for each* $I \subseteq [c]$*,*
(U3) *there are at least* 3 *vertices* v *with* $d_{v,I} \geq 2$ *for each critical set* $I \subseteq [c]$*, and*
(U4) *there are no two disjoint critical sets.*

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