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The pseudograph (r, s, a, t)-threshold number

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ABSTRACT

For $d \geq 1, s \geq 0$, a *$(d, d+s)$-graph* is a graph whose degrees all lie in the interval $\{d, d+1, \ldots, d+s\}$. For $r \geq 1, a \geq 0$, an *$(r, r+a)$-factor* of a graph G is a spanning $(r, r+a)$-subgraph of G. An *$(r, r+a)$-factorization* of a graph G is a decomposition of G into edge-disjoint $(r, r+a)$-factors. A *pseudograph* is a graph which may have multiple edges and may have multiple loops. A loop counts two towards the degree of the vertex it is on. A *multigraph* here has no loops.

For $t \geq 1$, let $\pi(r, s, a, t)$ be the least integer such that, if $d \geq \pi(r, s, a, t)$ then every $(d, d+s)$-pseudograph G has an $(r, r+a)$-factorization into $x(r, r+a)$-factors for at least t different values of x. We call $\pi(r, s, a, t)$ the *pseudograph (r, s, a, t)-threshold number*. Let $\mu(r, s, a, t)$ be the analogous integer for multigraphs. We call $\mu(r, s, a, t)$ the *multigraph (r, s, a, t)-threshold number*. A *simple graph* has at most one edge between any two vertices and has no loops. We let $\sigma(r, s, a, t)$ be the analogous integer for simple graphs. We call $\sigma(r, s, a, t)$ the *simple graph (r, s, a, t)-threshold number*.

In this paper we give the precise value of the pseudograph $\pi(r, s, a, t)$-threshold number for each value of r, s, a and t. We also use this to give good bounds for the analogous simple graph and multigraph threshold numbers $\sigma(r, s, a, t)$ and $\mu(r, s, a, t)$.

1. Introduction

In this paper a *pseudograph* is a graph which may have multiple edges and multiple loops, and a loop counts two towards the degree of the vertex it is on. A *multigraph* may have multiple edges, but has no loops. A *simple graph* has just simple edges and no loop. We use the term graph when a general statement is being made which applies to any of simple graph, multigraph or pseudograph.

It is well-known that in general a d-regular graph need not have a factorization into r-regular factors when $r \mid d$. For example, the Petersen graph cannot be factorized into three 1-factors. But if d is large enough, a d-regular simple graph will have a factorization into $(r, r+1)$-semiregular factors, i.e. factors in which the degree of each vertex lies in the set $\{r, r+1\}$.

For $d \geq 1, s \geq 0$, a $(d, d+s)$-graph is a graph whose degrees all lie in the set $\{d, d+1, \ldots, d+s\}$. For $r \geq 1, a \geq 0$ an $(r, r+a)$-factor of a graph G is a spanning $(r, r+a)$- subgraph of G. An $(r, r+a)$-factorization of a graph G is a decomposition of G into edge-disjoint $(r, r+a)$-factors.

Let $\beta(r, s, a, t)$ be the least integer such that, if $d \geq \beta(r, s, a, t)$, then every $(d, d+s)$-bipartite multigraph G has an $(r, r+a)$-factorization into $x(r, r+a)$-factors for at least t different values of x. We call $\beta(r, s, a, t)$ the bipartite (r, s, a, t)-threshold number. If we restrict attention to bipartite simple graphs, then the corresponding number is denoted by $\beta_s(r, s, a, t)$ and called the simple bipartite (r, s, a, t)-threshold number.

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The simple graph (r, s, a, t)-threshold number, $\sigma(r, s, a, t)$, the multigraph (r, s, a, t)-threshold number, $\mu(r, s, a, t)$, and the pseudograph (r, s, a, t)-threshold number, $\pi(r, s, a, t)$, are defined similarly. For example $\pi(r, s, a, t)$ is the least integer such that, if $d \geq \pi(r, s, a, t)$ then every $(d, d+s)$-pseudograph G has an $(r, r+a)$-factorization into $x(r, r+a)$-factors for at least t different values of x.

Functions with four parameters take some getting used to. The strangest is the parameter t. It may help to think of a regular simple graph of degree 29. It is shown in [6] that any such graph has a $(2, 3)$-factorization with $x(2, 3)$-factors for each $x \in \{10, 11, 12, 13, 14\}$, but for no other values of x.

To date, the only threshold number which has been completely determined is the bipartite threshold number $\beta(r, s, a, t)$, valid for bipartite multigraphs. Here we add to this list of completely determined threshold numbers by evaluating the pseudograph threshold numbers $\pi(r, s, a, t)$ and the simple bipartite threshold number $\beta_s(r, s, a, t)$.

It will be convenient to define the function $N(r, s, a, t)$ as follows.

Definition 1.1. Let r and t be positive integers, and let a and s be non-negative integers. Then

$$N(r, s, a, t) = r\left\lceil \frac{tr + s - 1}{a} \right\rceil + (t - 1)r.$$

The following result was proved in [7].

Proposition 1.2. *Let r and t be positive integers, and let a and s be non-negative integers. Then*

$$\beta(r, s, a, t) = N(r, s, a, t).$$

Perhaps it may seem surprising that a 4-parameter function like this can be evaluated. However, the proof of this result was relatively simple. To evaluate $\pi(r, s, a, t)$, the threshold number for pseudographs, one of the main results of this paper, has been by contrast a considerable struggle, and it surprised us that we were able finally to determine it.

We shall reserve the word Theorem for a statement that is at least partly new, and the word Proposition for a statement of interest but proved elsewhere. The word Lemma will be reserved for results, whether new or old, that are used in the proofs of our main results.

Our first main result contains our evaluation of $\beta_s(r, s, a, t)$. The statement of Theorem 1.3 includes an earlier result given below as Lemma 1.10.

Theorem 1.3. *Let r, a and t be positive integers, and let s be a non-negative integer. Then*

$$\beta_s(r, s, a, t) = N(r, s, a, t) \leq \sigma(r, s, a, t) \leq \mu(r, s, a, t) \leq \pi(r, s, a, t).$$

Our second main result is our evaluation of $\pi(r, s, a, t)$.

Theorem 1.4. *Let r, s, a and t be integers with r and t positive, $a \geq 2$ and s non-negative.*

(1) *If r and a are both even, then*

$$\pi(r, s, a, t) = N(r, s, a, t).$$

(2) *If r and a are both odd, then*

$$\pi(r, s, a, t) = \begin{cases} N(r+1, s, a-1, t) - 1 & \text{if } (r+1)t + s \not\equiv 2 (\text{mod } a-1), \\ N(r+1, s, a-1, t) - (r+1) - 1 & \text{if } (r+1)t + s \equiv 2 (\text{mod } a-1). \end{cases}$$

(3) *If r is odd and a is even, then*

$$\pi(r, s, a, t) = \begin{cases} N(r+1, s, a-2, t) - 1 & \text{if } (r+1)t + s \not\equiv 2 (\text{mod } a-2), \\ N(r+1, s, a-2, t) - (r+1) - 1 & \text{if } (r+1)t + s \equiv 2 (\text{mod } a-2). \end{cases}$$

(4) *If r is even and a is odd, then*

$$\pi(r, s, a, t) = \begin{cases} N(r, s, a-1, t) & \text{if } rt + s \not\equiv 2 (\text{mod } a-1), \\ N(r, s, a-1, t) - r & \text{if } rt + s \equiv 2 (\text{mod } a-1). \end{cases}$$

For $a = 0$ or 1, we give the results in Theorem 1.5. Note that we use the notation $\pi(r, s, a, t) = \infty$ when there is no finite threshold number for the given value of r, s, a and t.