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A $\frac{9}{7}$-approximation algorithm for Graphic TSP in cubic bipartite graphs[☆]

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ABSTRACT

We prove new results for approximating the Graphic TSP. Specifically, we provide a polynomial-time $\frac{9}{7}$-approximation algorithm for cubic bipartite graphs and a $(\frac{9}{7}+\frac{1}{21(k-2)})$-approximation algorithm for k-regular bipartite graphs, both of which are improved approximation factors compared to previous results. Our approach involves finding a cycle cover with relatively few cycles, which we are able to do by leveraging the fact that all cycles in bipartite graphs are of even length along with our knowledge of the structure of cubic graphs.

1. Introduction

1.1. Motivation and related work

The traveling salesman problem (TSP) is one of most well known problems in combinatorial optimization, famous for being hard to solve precisely. In this problem, given a complete undirected graph $G = (V, E)$ with vertex set V and edge set E, with non-negative edge costs $c \in \mathbb{R}^{|E|}$, $c \neq 0$, the objective is to find a Hamiltonian cycle in G of minimum cost. In its most general form, TSP cannot be approximated in polynomial time unless $P = NP$. In order to successfully find approximate solutions for TSP, it is common to require that instances of the problem have costs that satisfy the triangle inequality ($c_{ij} + c_{jk} \geq c_{ik}\ \forall i, j, k \in V$). This is the Metric TSP. The Graphic TSP is a special case of the Metric TSP, where instances are restricted to those where $\forall i, j \in E$, the cost of edge (i, j) in the complete graph G are the lengths of the shortest paths between nodes i and j in an unweighted, undirected graph, on the same vertex set.

One value related to the ability to approximate TSP is the integrality gap, which is the worst-case ratio between the optimal solution for a TSP instance and the solution to a linear programming relaxation called the subtour relaxation [6]. A long-standing conjecture (see, e.g.,[10]) for Metric TSP is that the integrality gap is $\frac{4}{3}$. One source of motivation for studying the Graphic TSP is that the family of graphs with two vertices connected by three paths of length k has an integrality gap that approaches $\frac{4}{3}$. This family of graphs demonstrates that The Graphic TSP captures much of the complexity of the more general Metric TSP.

For several decades, the Graphic TSP did not have any approximation algorithms that achieved a better approximation than Christofides' classic $\frac{3}{2}$-approximation algorithm for Metric TSP [4], further motivating the study of this problem. However, a wave of recent papers [8,1,3,9,12,5,14] have provided significant improvements in approximating the Graphic

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TSP. Currently, the best known approximation algorithm for the Graphic TSP is due to Sebő and Vygen [14], with an approximation factor of $\frac{7}{5}$.

Algorithms with even smaller approximation factors have also been found for the Graphic TSP instances generated by specific subclasses of graphs. In particular, algorithms for the Graphic TSP in cubic graphs (where all nodes have degree 3) have drawn significant interest as this appears to be the simplest class of graphs that has many of the same challenges as the general case. Currently, the best approximation algorithm for the Graphic TSP in cubic graphs is due to Correa, Larré, and Soto [5], whose algorithm achieves an approximation factor of $(\frac{4}{3} - \frac{1}{61236})$ for 2-edge-connected cubic graphs. Similarly, a $\frac{4}{3}$-approximation was recently obtained for instances of sub-quartic graphs [13]. In a recent preprint, van Zuylen [15] extends the work in [5,12] to obtain a $\frac{5}{4}$-approximation algorithm for cubic bipartite graphs and a $(\frac{4}{3} - \frac{1}{8754})$-approximation algorithm for 2-connected cubic graphs.

Progress in approximating the Graphic TSP in cubic graphs also relates to traditional graph theory, as Barnette's conjecture [2] states that all bipartite, planar, 3-connected, cubic graphs are Hamiltonian. This conjecture suggests that instances of Graph TSP on Barnette graphs could be easier to approximate, and conversely, approximation algorithms for the Graphic TSP in Barnette graphs may lead to the resolution of this conjecture. Indeed, Correa, Larré, and Soto [5] provided a $(\frac{4}{3} - \frac{1}{18})$-approximation algorithm for Barnette graphs. Along these lines, Aggarwal, Garg, and Gupta [1] were able to obtain a $\frac{4}{3}$-approximation algorithm for 3-edge-connected cubic graphs before any $\frac{4}{3}$-approximation algorithms were known for all cubic graphs. In this paper, we examined graphs that are cubic and bipartite, another class of graphs that includes all Barnette graphs. An improved approximation for this class of graphs is the primary theoretical contribution of this paper.

Theorem 1.1. *Given a cubic bipartite connected graph G with n vertices, there is a polynomial time algorithm that computes a spanning Eulerian multigraph H in G with at most $\frac{9}{7}n$ edges.*

Corollary 1.2. *Given a k-regular bipartite connected graph G with n vertices where $k \geq 4$, there is a polynomial time algorithm that computes a spanning Eulerian multigraph H in G with at most $(\frac{9}{7} + \frac{1}{21(k-2)})n - 2$ edges.*

This extension complements results [16,7] which provide guarantees for k-regular graphs in the asymptotic regime. Corollary 1.2 improves on these guarantees for small values of k. Note that even for $k = 4$ Corollary 1.2 yields a solution with fewer then $\frac{4}{3}n$ edges.

1.2. Overview

In this paper, we will present an algorithm to solve the Graphic TSP, which guarantees a solution with at most $\frac{9}{7}n$ edges in cubic bipartite graphs. The best possible solution to the Graphic TSP is a Hamiltonian cycle, which has exactly n edges, so this algorithm has an approximation factor of $\frac{9}{7}$.

A corollary of Petersen's theorem is that every cubic bipartite graph contains three edge-disjoint perfect matchings. The union of any 2 of these matchings forms a 2-factor. The following proposition demonstrates the close relationship between 2-factors and Graphic TSP tours in connected graphs.

Proposition 1.3. *Any 2-factor with k cycles in a connected graph can be extended into a spanning Eulerian multigraph with the addition of exactly $2(k-1)$ edges. This multigraph contains exactly $n + 2(k-1)$ edges in total.*

Proposition 1.3 can be implemented algorithmically by compressing each cycle in the 2-factor into a single node and then finding a spanning tree in this compressed graph. We then add two copies of the edges from this spanning tree to the 2-factor. We present an algorithm, BIGCYCLE, which begins by finding a 2-factor with at most $\frac{n}{7}$ cycles. Then, it applies Proposition 1.3 to generate a spanning Eulerian subgraph from this 2-factor containing at most $n + 2 \times (\frac{n}{7} - 1) = \frac{9}{7}n - 2$ edges.

BIGCYCLE first shrinks 4-cycles in the graph, then it generates a 2-factor in the condensed graph. If the resulting 2-factor has no 4- or 6-cycles, then we can expand the 4-cycles and this will be our solution. If the 2-factor has a 4-cycle, then we either contract it or show that this cycle will be extended into a longer cycle when we expand the graph. If the 2-factor does have a 6-cycle, then the algorithm contracts either this 6-cycle or a larger subgraph that includes this 6-cycle. We are able to iterate this process until we find a 2-factor in the compressed graph that can be shown to expand into a 2-factor in the original graph with relatively few cycles. Theorem 3.12 in Section 3.5 proves that this 2-factor has at most $\frac{n}{7}$ cycles.

2. A $\frac{9}{7}$-approximation algorithm for graphic TSP in cubic bipartite graphs

2.1. Overview

In a graph with no 4-cycles (squares), all 2-factors will have an average cycle length of at least 6, so all 2-factors will have at most $\frac{n}{6}$ cycles, which results in a $\frac{4}{3}$-approximation after applying Proposition 1.3. In order to improve our approximation guarantee, we need to target 6-cycles, as well as 4-cycles. The algorithm we present finds a square-free 2-factor in which every 6-cycle can be put in correspondence with a distinct cycle of size 8 or larger. Then, we can find a 2-factor in which