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# Weighted independent sets in classes of $P_6$-free graphs

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ABSTRACT

The Maximum Weight Independent Set (MWIS) problem on graphs with vertex weights asks for a set of pairwise nonadjacent vertices of maximum total weight. The complexity of the MWIS problem for $P_6$-free graphs, and for $S_{1,2,2}$-free graphs are unknown. In this note, we give a proof for the solvability of the MWIS problem for ($P_6$, $S_{1,2,2}$, co-chair)-free graphs in polynomial time, by analyzing the structure and the MWIS problem in various subclasses of ($P_6$, $S_{1,2,2}$, co-chair)-free graphs. These results extend some known results in the literature.



## 1. Introduction

In a graph $G$, an *independent (or stable) set* is a subset of mutually nonadjacent vertices in $G$. The Maximum Independent Set (MIS) problem asks for an independent set of $G$ with maximum cardinality. The Maximum Weight Independent Set (MWIS) problem asks for an independent set of total maximum weight in the given graph $G$ with vertex weight function $w$ on $V(G)$. The M(W)IS problem ([GT20] in [19]) is one of the fundamental algorithmic graph problems; it frequently occurs as a subproblem in models in computer science, bioinformatics, operations research and other fields. Also, the problem has numerous applications, including train dispatching [17] and data mining [37]. The M(W)IS problem is well known to be NP-complete in general and hard to approximate; it remains NP-complete even on restricted classes of graphs.

If $\mathcal{F}$ is a family of graphs, we say that a graph $G$ is $\mathcal{F}$*-free* if it contains no induced subgraph isomorphic to any member of $\mathcal{F}$. Let $K_n$, $P_n$ and $C_n$ denote respectively the complete graph, the path, and the cycle on $n$ vertices. Given a graph $F$, let $kF$ denote the disjoint union of $k$ copies of $F$. Alekseev [1] proved that the M(W)IS problem remains NP-complete on $H$-free graphs whenever $H$ is connected, but neither a path nor a subdivision of the claw ($K_{1,3}$). On the other hand, the M(W)IS problem is known to be solvable in polynomial time on many graph classes such as: perfect graphs [21]; $2K_2$-free graphs [16]; claw-free graphs [33]; and fork-free graphs [30].

Here we focus on graphs without long induced paths and a subdivision of a claw. For integers $i, j, k \geq 1$, let $S_{i,j,k}$ denote a tree with exactly three vertices of degree one, being at distance $i, j$ and $k$ from the unique vertex of degree three. The graph $S_{1,1,1}$ is called a *claw* and $S_{1,1,2}$ is called a *chair or fork*. Also, note that $S_{i,j,k}$ is a subdivision of a claw.

A *diamond* is a graph with vertex set $\{a, b, c, d\}$ and edge set $\{ab, ac, bc, bd, cd\}$. A *gem* is a graph with vertex set $\{a, b, c, d, e\}$ and edge set $\{ab, bc, cd, ae, be, ce, de\}$. A *house* is a graph that is isomorphic to the complement of $P_5$. See Fig. 1 for some special graphs that we use in this paper.

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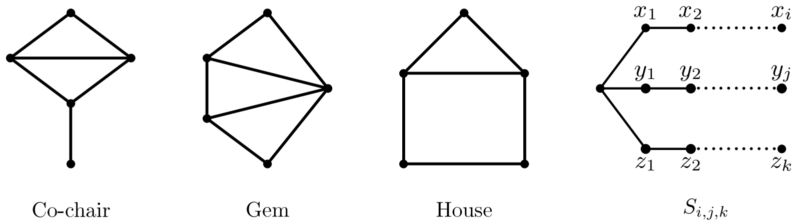


**Fig. 1.** Some special graphs.

As a natural generalization of the well studied $P_4$-free graphs, $2K_2$-free graphs and semi-$P_4$-sparse graphs (being the class of ($P_5$, house, co-chair)-free graphs [18]), the class of $P_k$-free graphs ($k \geq 5$), the class of $S_{1,j,k}$-free graphs ($j, k \geq 2$), and the class of $S_{1,1,k}$-free graphs ($k \geq 3$) have received attention. The complexity of the MWIS problem for $P_5$-free graphs was unknown for several decades, though the problem is shown to be solvable in polynomial time in various subclasses of $P_5$-free graphs [5,25]. Recently, Lokshtanov, Vatshelle and Villanger [29] gave a $O(n^{12}m)$ time algorithm for the MWIS problem in $P_5$-free graphs via minimal triangulations, and they showed an $O(n^{18}m)$ time algorithm for the MWIS problem on $P_5$-free graphs. However, the complexity of the MWIS problem is unknown for the class of $P_6$-free graphs, for the class of $S_{1,2,2}$-free graphs, and for the class of $S_{1,1,3}$-free graphs. In particular, the class of $P_6$-free graphs, the class of $S_{1,2,2}$-free graphs, and the class of $S_{1,1,3}$-free graphs constitute the minimal class, defined by forbidding a single connected subgraph on six vertices, for which the computational complexity of M(W)IS problem is unknown. Recently, Lokshtanov, Pilipczuk and van Leeuwen [28] showed that there is an $n^{O(\log^2 n)}$-time, polynomial-space algorithm for MWIS on $P_6$-free graphs. This implies that MWIS on $P_6$-free graphs is not *NP*-complete, unless all problems in *NP* can be solved in quasi-polynomial time. On the other hand, the M(W)IS problem can be efficiently solved in many subclasses of $P_6$-free graphs, subclasses of $S_{1,2,2}$-free graphs, and subclasses of $S_{1,1,3}$-free graphs: ($P_6$, triangle)-free graphs [10], ($P_6$, $K_{1,p}$)-free graphs [31], ($P_6$, $C_4$)-free [9,26,34], ($P_6$, diamond)-free graphs [35], ($P_6$, banner)-free graphs [26], ($P_6$, co-banner)-free graphs [36], ($S_{1,2,2}$, banner)-free graphs [20], ($S_{1,1,3}$, banner)-free graphs [27], and ($S_{1,2,2}$, bull)-free graphs [27]. Note that the class of $P_5$-free graphs is a subclass of $P_6$-free graphs, $S_{1,2,2}$-free graphs, and $S_{1,1,3}$-free graphs. Also, the class of fork-free graphs is a subclass of $S_{1,2,2}$-free graphs, and $S_{1,1,3}$-free graphs.

Graph decompositions such as clique separator decomposition and modular decomposition (defined below) play a crucial role in structural graph theory and in designing efficient graph algorithms. A *clique* in a graph $G$ is a subset of pairwise adjacent vertices in $G$. A *clique separator* of a graph $G$ is a clique $K$ in $G$ such that $G \setminus K$ has more connected components than $G$. An *atom* is a graph without clique separator. A *clique separator decomposition (CSD)* of a graph $G$ is a decomposition tree of $G$ whose leaves are atoms. Whitesides [39] proved that CSD of a graph can be obtained in polynomial time; Tarjan [38] showed that CSD can be applied to various optimization problems such as Minimum Fill-In, Coloring, Maximum Clique, and the MWIS problem; the problem can be solved efficiently on the graph if it is solvable efficiently on atoms. Using this approach, it is shown that MWIS is solvable in polynomial time for several classes of graphs recently; see [2–4,8,11,13,25].

We follow the approach developed recently by Brandstädt and Giakoumakis [8], which combines modular decomposition [32] and clique separator decomposition, and prove that the MWIS problem can be solved in polynomial time in the class of ($P_6$, co-chair, $S_{1,2,2}$)-free graphs by analyzing the atomic structure and the MWIS problem in several subclasses of that class. These results extend the following known results: the aforementioned results for $P_4$-free graphs, semi-$P_4$-sparse graphs [18], ($P_5$, diamond)-free graphs [6], and ($P_5$, co-chair)-free graphs [14,25].

## 2. Notation, terminology and preliminaries

For notations and terminology, we follow [12]. Let $G$ be a finite, undirected and simple graph with vertex-set $V(G)$ and edge-set $E(G)$. We let $|V(G)| = n$ and $|E(G)| = m$. If $S \subseteq V(G)$, then $G[S]$ denote the subgraph induced by $S$ in $G$. For a vertex $v \in V(G)$, the *neighborhood* $N(v)$ of $v$ is the set $\{u \in V(G) \mid uv \in E(G)\}$. The neighborhood $N(X)$ of a subset $X \subseteq V(G)$ is the set $\{u \in V(G) \setminus X : u$ is adjacent to a vertex of $X\}$. Given a subgraph $H$ of $G$ and $v \in V(G) \setminus V(H)$, let $N_H(v)$ denote the set $N(v) \cap V(H)$, and for $X \subseteq V(G) \setminus V(H)$, let $N_H(X)$ denote the set $N(X) \cap V(H)$. For any two subsets $S, T \subseteq V(G)$, we say that $S$ is *complete* to $T$ if every vertex in $S$ is adjacent to every vertex in $T$. We say that $S$ is *anticomplete* to $T$ if there is no edge between $S$ and $T$.

### *Homogeneous sets and modular decomposition*

A vertex $z \in V(G)$ *distinguishes* two other vertices $x, y \in V(G)$ if $z$ is adjacent to one of them and not adjacent to the other. A set $M \subseteq V(G)$ is a *homogeneous set* in $G$ if no vertex from $V(G) \setminus M$ distinguishes two vertices from $M$. The *trivial* homogeneous sets of $G$ are $V(G)$, $\emptyset$, and all one-elementary vertex sets. A graph $G$ is *prime* if it has only trivial homogeneous sets. Note that prime graphs are connected. We will use the following theorem by Lozin and Milanič [30].