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The optimal rubbling number of ladders, prisms and Möbius-ladders

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ABSTRACT

A pebbling move on a graph removes two pebbles at a vertex and adds one pebble at an adjacent vertex. Rubbling is a version of pebbling where an additional move is allowed. In this new move, one pebble each is removed at vertices v and w adjacent to a vertex u, and an extra pebble is added at vertex u. A vertex is reachable from a pebble distribution if it is possible to move a pebble to that vertex using rubbling moves. The optimal rubbling number is the smallest number m needed to guarantee a pebble distribution of m pebbles from which any vertex is reachable. We determine the optimal rubbling number of ladders ($P_n \square P_2$), prisms ($C_n \square P_2$) and Möbius-ladders.

1. Introduction

Graph pebbling has its origin in number theory. It is a model for the transportation of resources. Starting with a pebble distribution on the vertices of a simple connected graph, a *pebbling move* removes two pebbles from a vertex and adds one pebble at an adjacent vertex. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. A vertex is called *reachable* if a pebble can be moved to that vertex using pebbling moves. There are several questions we can ask about pebbling. One of them is: How can we place the smallest number of pebbles such that every vertex is reachable (*optimal pebbling number*)? For a comprehensive list of references for the extensive literature see the survey papers [4–6].

Graph rubbling is an extension of graph pebbling. In this version, we also allow a move that removes a pebble each from the vertices v and w that are adjacent to a vertex u, and adds a pebble at vertex u. The basic theory of rubbling and optimal rubbling is developed in [1]. The rubbling number of complete m-ary trees are studied in [3], while the rubbling number of caterpillars are determined in [9]. In [7] the authors give upper and lower bounds for the rubbling number of diameter 2 graphs.

In the present paper we determine the optimal rubbling number of ladders ($P_n \square P_2$), prisms ($C_n \square P_2$) and Möbius-ladders.

2. Definitions

Throughout the paper, let G be a simple connected graph. We use the notation $V(G)$ for the vertex set and $E(G)$ for the edge set. A *pebble function* on a graph G is a function $p : V(G) \to \mathbb{Z}$ where $p(v)$ is the number of pebbles placed at v. A *pebble*

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distribution is a nonnegative pebble function. The *size* of a pebble distribution p is the total number of pebbles $\sum_{v\in V(G)} p(v)$. If H is a subgraph of G, then $p(H) = \sum_{v\in V(H)} p(v)$. We say that a vertex v is *occupied* if $p(v) > 1$, else it is *unoccupied*.

Consider a pebble function p on the graph G. If $\{v, u\} \in E(G)$ then the *pebbling move* $(v, v \to u)$ removes two pebbles at vertex v, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 2$ and $p'(u) = p(u) + 1$. If $\{w, u\} \in E(G)$ and $v \neq w$, then the *strict rubbling move* $(v, w \to u)$ removes one pebble each at vertices v and w, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 1$, $p'(w) = p(w) - 1$ and $p'(u) = p(u) + 1$.

A *rubbling move is* either a pebbling move or a strict rubbling move. A *rubbling sequence* is a finite sequence $T = (t_1, \ldots, t_k)$ of rubbling moves. The pebble function obtained from the pebble function p after applying the moves in T is denoted by p_T. The concatenation of the rubbling sequences $R = (r_1, \ldots, r_k)$ and $S = (s_1, \ldots, s_l)$ is denoted by $RS = (r_1, \ldots, r_k, s_1, \ldots, s_l)$.

A rubbling sequence T is *executable* from the pebble distribution p if $p_{(t_1,\ldots,t_i)}$ is nonnegative for all i. A vertex v of G is *reachable* from the pebble distribution p if there is an executable rubbling sequence T such that $p_T(v) \geq 1$. p is a solvable distribution when each vertex is reachable. Correspondingly, v is *k-reachable* under p if there is an executable T, that $p_T(v) \geq k$, and p is *k-solvable* when every vertex is k-reachable. An H subgraph is *k-reachable* if there is an executable rubbling sequence T such that $p_T(H) = \sum_{v\in V(H)} p_T(v) \geq k$. We say that vertices u and v are *independently reachable* if there is an executable rubbling sequence T such that $p_T(u) \geq 1$ and $p_T(v) \geq 1$.

The *optimal rubbling number* $\varrho_{\text{opt}}(G)$ of a graph G is the size of a distribution with the least number of pebbles from which every vertex is reachable. A solvable pebbling distribution is *optimal* if its size equals to the optimal rubbling number.

Let G and H be simple graphs. Then the *Cartesian product* of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if and only if $g = g'$ and $(h, h') \in E(H)$ or if $h = h'$ and $(g, g') \in E(G)$. This graph is denoted by $G\square H$.

P_n and C_n denotes the path and the cycle containing n distinct vertices, respectively. We call $P_n\square P_2$ a *ladder* and $C_n\square P_2$ a *prism*. It is clear that the prism can be obtained from the ladder by joining the 4 endvertices by two edges to form two vertex disjoint C_n subgraphs. If the four endvertices are joined by two new edges in a switched way to get a C_{2n} subgraph, then a *Möbius-ladder* is obtained.

We imagine the $P_n\square P_2$ ladder laid horizontally, so there is an upper P_n path, and a lower P_n path, which are connected by "parallel" edges, called *rungs* of the ladder. Vertices on the upper path will be usually denoted by v_i, while vertices of the lower path by w_i. Also, if A is a rung (a vertical edge of the graph), then $\overline{A}$ denotes the upper, and $\underline{A}$ the lower endvertex of this rung. This arrangement also defines a natural left and right direction on the horizontal paths, and between the rungs.

3. Optimal rubbling number of the ladder

In this section we give a formula for the optimal rubbling number of ladders:

Theorem 3.1. *Let $n = 3k + r$ such that $0 \leq r < 3$ and $n, r \in \mathbb{N}$, so $k = \left\lfloor \frac{n}{3} \right\rfloor$. Then*

$$\varrho_{\text{opt}}(P_n\square P_2) = \begin{cases} 1 + 2k & \text{if } r = 0, \\ 2 + 2k & \text{if } r = 1, \\ 2 + 2k & \text{if } r = 2. \end{cases}$$

In the rest of the section we are going to prove the above theorem. The proof is fairly long and complex, so it will be divided to several lemmas. First, we prove that the function given in the theorem is an upper bound, by giving solvable distributions.

Lemma 3.2. *Let $n = 3k + r$ such that $0 \leq r < 3$ and $n, r \in \mathbb{N}$, so $k = \left\lfloor \frac{n}{3} \right\rfloor$.*

$$\varrho_{\text{opt}}(P_n\square P_2) \leq \begin{cases} 1 + 2k & \text{if } r = 0, \\ 2 + 2k & \text{if } r = 1, \\ 2 + 2k & \text{if } r = 2. \end{cases}$$

Proof. A solvable distribution with adequate size is shown in Fig. 1 for each case. □

Now we need to prove that the function is a lower bound as well. This part is unfortunately much harder. Before we start the rigorous proof, a summary of the proof is given. Then the necessary definitions and proofs of several Lemmas will follow.

Summary of the proof: We prove by induction on n. First we deal with the base cases in Lemma 3.3. For the induction step, consider an optimal distribution p on $P_n\square P_2$. Choose an appropriate $R = P_3\square P_2$ subgraph which contains maximum number of pebbles, delete the vertices of R and reconnect the remaining two parts to obtain $G^R = P_{n-3}\square P_2$, called the *reduced graph*, see Fig. 2.

Now construct a solvable p^R distribution for the new $P_{n-3}\square P_2$ graph in the following way: p induces a distribution on the vertices which we have not deleted. In most of the cases we simply place $p(v)$ pebbles to all $v \in V(G) \setminus V(R)$, (i.e. do not change the original distribution), in some other cases we apply a simple operation on the original distribution. Finally,