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An extension of Lehman's theorem and ideal set functions

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ABSTRACT

Lehman's theorem on the structure of minimally nonideal clutters is a fundamental result in polyhedral combinatorics. One approach to extending it has been to give a common generalization with the characterization of minimally imperfect clutters (Sebö, 1998; Gasparyan et al., 2003). We give a new generalization of this kind, which combines two types of covering inequalities and works well with the natural definition of minors. We also show how to extend the notion of idealness to unit-increasing set functions, in a way that is compatible with minors and blocking operations.

1. Introduction

A set family $\mathcal{C}$ on a ground set V of size n is called a *clutter* if no set in $\mathcal{C}$ is a subset of another. We will refer to elements of V simply as *elements*, while elements of $\mathcal{C}$ will be referred to as *members* of $\mathcal{C}$. Let $\mathcal{C}^\uparrow$ denote the uphull of $\mathcal{C}$, that is, $\mathcal{C}^\uparrow = \{U \subseteq V : U \supseteq C \text{ for some } C \in \mathcal{C}\}$. The *blocker* $b(\mathcal{C})$ of a clutter $\mathcal{C}$ is defined as the family of the (inclusionwise) minimal sets that intersect each member of $\mathcal{C}$. It is easy to check that $b(b(\mathcal{C})) = \mathcal{C}$, see e.g. [3, Theorem 1.3].

One of the most well-studied objects of polyhedral combinatorics is the *covering polyhedron* of a clutter, which we consider in the following bounded version:

$$P(\mathcal{C}) = \{x \in \mathbb{R}^V : \mathbf{0} \le x \le \mathbf{1},\ x(C) \ge 1 \text{ for every } C \in \mathcal{C}\},$$

where $x(C)$ denotes $\sum_{v \in C} x_v$. The integer points of $P(\mathcal{C})$ correspond to the sets in $b(\mathcal{C})^\uparrow$. A clutter $\mathcal{C}$ is called *ideal* if the polyhedron $P(\mathcal{C})$ is integer. By a result of Lehman [9], a clutter is ideal if and only if its blocker is.

Deciding whether a clutter is ideal is hard (see e.g. [6], where it is shown to include the co-NP-complete problem of recognizing quasi-bipartite graphs). However, interesting structural properties can be proved for clutters which are minimally nonideal (*mni*) in the sense that any facet of P defined by setting a variable to 0 or 1 is integer. A simple infinite family of mni clutters is the family of finite degenerate projective planes, defined as $\mathcal{J}_t = \{\{1, 2, \ldots, t\}, \{0, 1\}, \{0, 2\}, \ldots \{0, t\}\}$ on ground set $\{0, 1, \ldots, t\}$, where $t \ge 2$. It is easy to check that the blocker of $\mathcal{J}_t$ is itself. The following theorem of Lehman [9,10], which shows that all other mni clutters have a regular structure, is considered to be one of the fundamental results on covering polyhedra.

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Theorem 1.1 (*Lehman [9,10]*)**.** *Let $\mathcal{C}$ be a minimally nonideal clutter nonisomorphic to a finite degenerate projective plane. Then $P(\mathcal{C})$ has a unique noninteger vertex, namely $\frac{1}{r}\mathbf{1}$, where r is the minimum size of an edge in $\mathcal{C}$. There are exactly n sets of size r in $\mathcal{C}$ and each element of V is contained in exactly r of them. The blocker $b(\mathcal{C})$ also has exactly n sets of minimum size, which correspond to the vertices of $P(\mathcal{C})$ adjacent to the noninteger vertex.*

An important consequence of the theorem, observed by Seymour [15], is that the problem of deciding idealness of a clutter is in co-NP, provided that we have a membership oracle for $\mathcal{C}^{\uparrow}$. After Lehman's groundbreaking result, there have been several attempts to better understand the structure of minimally nonideal clutters (see [12] for an enumeration of mni matrices of small dimension, [5] for a characterization of mni circulants, [3] for a survey, and [4,16] for more recent developments).

There have been successful efforts to combine Lehman's theorem with another fundamental result, the co-NP characterization of minimally imperfect clutters by Lovász and Padberg [11,13]. A clutter $\mathcal{D}$ is *perfect* if the packing polyhedron $\{x \in \mathbb{R}^V : \mathbf{0} \le x \le \mathbf{1},\ x(D) \le 1 \text{ for every } D \in \mathcal{D}\}$ is integral. It is minimally imperfect if it is not perfect, but any face of the polyhedron obtained by setting some variable to 0 is integral. Note that it is unnecessary to consider faces obtained by setting a variable to 1, because if the face $x_v = 0$ is integral, then so is the face $x_v = 1$.

Theorem 1.2 (*Lovász [11], Padberg [13]*)**.** *If a clutter $\mathcal{D}$ is minimally imperfect, then it either consists of all $(n-1)$-element subsets of V (the non-Helly clutter), or it consists of the maximal cliques of a minimally imperfect graph. In both cases, $\mathcal{D}$ has n maximum size members, and they form a regular hypergraph.*

Of course, we can claim much stronger properties for $\mathcal{D}$ using the Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour, and Thomas [2].

Theorem 1.3 (*Strong Perfect Graph Theorem [2]*)**.** *A graph is perfect if and only if it contains no odd hole (an induced subgraph isomorphic to an odd cycle of length at least 5) and no odd antihole (an induced subgraph isomorphic to the complement of an odd cycle of length at least 5).*

It follows that if $\mathcal{D}$ is minimally imperfect, then it is either a non-Helly clutter, or a clutter formed by the inclusionwise maximal cliques of an odd cycle of length at least 5 or of the complement of an odd cycle of length at least 5.

Sebö [14], and Gasparyan, Preissmann and Sebö [8] considered polyhedra defined by both packing and covering constraints, and gave an extension of Lehman's theorem that includes Theorem 1.2. An inconvenience in their approach is that the class of polyhedra they consider is not closed under taking facets defined by setting variables to 0 or 1, and there is no natural way to define a blocker.

In this paper we present two different approaches that address these issues. In the first part of the paper, in Section 2, we prove an extension of Lehman's theorem to another class of polyhedra that includes both packing and covering polyhedra as a subclass. Let $\mathcal{C}$ and $\mathcal{D}$ be clutters on the same ground set V. We consider polyhedra of the form

$$P(\mathcal{C},\mathcal{D}) = \{x \in \mathbb{R}^V : \mathbf{0} \le x \le \mathbf{1},\ x(C) \ge 1 \text{ for every } C \in \mathcal{C},\ x(D) \ge |D|-1 \text{ for every } D \in \mathcal{D}\}.$$

If $\mathcal{D}$ is empty, then this is the same as $P(\mathcal{C})$. On the other hand, if $\mathcal{C}$ is empty, then $\{x \in \mathbb{R}^V : \mathbf{1}-x \in P(\mathcal{C},\mathcal{D})\}$ is the packing polyhedron of $\mathcal{D}$. Clearly, this polyhedron is integral if and only if $P(\mathcal{C},\mathcal{D})$ is integral. We will see that faces obtained by setting some variables to 0 or 1 are also polyhedra in this class, defined by appropriate pairs of clutters (these pairs will be called the minors of the pair $(\mathcal{C},\mathcal{D})$). Our main result is that if $P(\mathcal{C},\mathcal{D})$ is non-integral but the faces considered above are all integral, then one of the following holds: (a) $\mathcal{D}$ is empty and $\mathcal{C}$ is a minimally nonideal clutter, (b) $\mathcal{C}$ is empty and $\mathcal{D}$ is a minimally imperfect clutter, or (c) $\mathcal{D}$ has only members of size 2, and $\mathcal{C} \cup \mathcal{D}$ is an odd cycle or a degenerate projective plane.

As a corollary, we derive a new characterization of integrality of a polytope associated with the vertex cover problem in hypergraphs. Let $H = (V,\mathcal{E})$ be a hypergraph, and let G_H be the graph consisting of the hyperedges of size two in H. Lehman's theorem is a characterization of the integrality of the fractional vertex cover polyhedron for H. A weakness of this LP relaxation is that the polyhedron is automatically non-integer if G_H contains a triangle. To fix this, let us consider the polyhedron obtained by adding the clique inequalities of G_H:

$$P = \{x \in \mathbb{R}^V : \mathbf{0} \le x \le \mathbf{1},\ x(e) \ge 1 \text{ for every } e \in \mathcal{E},\ x(K) \ge |K|-1 \text{ for every clique } K \text{ in } G_H\}.$$

We give a Lehman-type characterization of the integrality of P. This implies that integrality is in co-NP even if the hypergraph is given implicitly by an oracle that outputs whether a given set $X \subseteq V$ induces a hyperedge or not.

An integer-valued set function f on ground set V is *unit-increasing* if $f(U) \le f(U+v) \le f(U)+1$ for every $U \subseteq V$ and $v \notin U$.[2] In the second part of the paper, in Section 3, we extend the notion of idealness to unit-increasing set functions. To a clutter $\mathcal{C}$ we can associate the unit-increasing function

$$f_{\mathcal{C}}(U) = \begin{cases} 1 & \text{if } U \in \mathcal{C}^{\uparrow}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

2 Throughout the paper, we use $U+v$ and $U-v$ as shorthand for $U \cup \{v\}$ and $U \setminus \{v\}$, respectively.