Discrete Applied Mathematics 209 (2016) 264–275

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Induced cycles in triangle graphs

Aparna Lakshmanan S [a], Csilla Bujtás [b,c,*], Zsolt Tuza [b,c]

[a] Department of Mathematics, St. Xavier's College for Women, Aluva, India
[b] Department of Computer Science and Systems Technology, University of Pannonia, Veszprém, Hungary
[c] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary

ARTICLE INFO

Article history:
Received 30 October 2014
Received in revised form 10 December 2015
Accepted 16 December 2015
Available online 13 January 2016

Keywords:
Triangle graph
F-free graph
Perfect graph
Tuza's Conjecture

ABSTRACT

The triangle graph of a graph G, denoted by $\mathcal{T}(G)$, is the graph whose vertices represent the triangles (K_3 subgraphs) of G, and two vertices of $\mathcal{T}(G)$ are adjacent if and only if the corresponding triangles share an edge. In this paper, we characterize graphs whose triangle graph is a cycle and then extend the result to obtain a characterization of C_n-free triangle graphs. As a consequence, we give a forbidden subgraph characterization of graph G for which $\mathcal{T}(G)$ is a tree, a chordal graph, or a perfect graph. For the class of graphs whose triangle graph is perfect, we verify a conjecture of the third author concerning packing and covering of triangles.

1. Introduction

In a simple undirected graph G, a *triangle* is a complete subgraph on three vertices. The *triangle graph* of G, denoted by $\mathcal{T}(G)$, is the graph whose vertices represent the triangles of G, and two vertices of $\mathcal{T}(G)$ are adjacent if and only if the corresponding triangles of G share an edge. This notion was introduced independently several times under different names and in different contexts [16,22,8,4]. One fundamental motivation is its obvious relation to the important class of line graphs.

In a more general setting, for a $k \geq 1$, the k-line graph $L_k(G)$ of G is a graph which has vertices corresponding to the K_k subgraphs of G, and two vertices are adjacent in $L_k(G)$ if the represented K_k subgraphs of G have $k-1$ vertices in common. Hence, 2-line graph means line graph in the usual sense, whilst 3-line graph is just the triangle graph, which is our current subject.

Beineke's classic result [5] gave a characterization of 2-line graphs in terms of nine forbidden subgraphs. This implies that 2-line graphs can be recognized in polynomial time. In contrast to this, as proved very recently in [2], the recognition problem of triangle graphs (and also, that of k-line graphs for each $k \geq 3$) is NP-complete. In the same paper [2], a necessary and sufficient condition is given for nontrivial connected graphs G and H to ensure that their Cartesian product $G \square H$ is a triangle graph.

Further related results have been obtained by Laskar, Mulder and Novick [11]. They prove that for an 'edge-triangular' and 'path-neighborhood' graph G (that is when the open neighborhood of v induces a non-trivial path for each vertex $v \in V(G)$), the triangle graph $\mathcal{T}(G)$ is a tree if and only if G is maximal outerplanar. Also, they raise the characterization problem of a path-neighborhood graph G for which $\mathcal{T}(G)$ is a cycle [11, Problem 3]. As an immediate consequence of our Theorem 4, we will answer this question; moreover we will give a forbidden subgraph characterization of graphs whose triangle graph is a tree.

Triangle graphs were studied from several further aspects; see e.g. [3,4,8,12–14,17–19].

* Corresponding author at: Department of Computer Science and Systems Technology, University of Pannonia, Veszprém, Hungary.
E-mail addresses: aparnaren@gmail.com (A. Lakshmanan S), bujtas@dcs.uni-pannon.hu (Cs. Bujtás), tuza@dcs.uni-pannon.hu (Zs. Tuza).

http://dx.doi.org/10.1016/j.dam.2015.12.012
0166-218X/

1.1. Standard definitions

Given a graph F, a graph G is called F*-free* if no *induced* subgraph of G is isomorphic to F. When $\mathcal{F}$ is a set of graphs, G is $\mathcal{F}$*-free* if it is F-free for all $F \in \mathcal{F}$. On the other hand, when we say that a graph F is a *forbidden subgraph* for a class $\mathcal{G}$ of graphs, it means that no $G \in \mathcal{G}$ may contain *any* subgraph isomorphic to F.

As usual, the complement of a graph G is denoted by $\overline{G}$. The nth power of a graph G is the graph G^n whose vertex set is $V(G^n) = V(G)$ and two vertices are adjacent in G^n if and only if their distance is at most n in G. Moreover, given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we use the notation $G_1 \vee G_2$ for the *join* of G_1 and G_2, that is a graph with one copy of G_1 and G_2 each, being vertex-disjoint, and all the vertices of G_1 are made adjacent with all the vertices of G_2. In particular, the n*-wheel* W_n ($n \geq 3$) is a graph $K_1 \vee C_n$ (where, as usual, K_n and C_n denote the n-vertex complete graph and the n-cycle, respectively). An *odd wheel* is a graph W_n where $n \geq 3$ is odd; and an *odd hole* in a graph is an *induced* n-cycle of odd length $n \geq 5$, whereas an *odd anti-hole* is the complement of an odd hole.

While an *acyclic* graph does not contain any cycles, a *chordal graph* is a graph which does not contain *induced* n-cycles for $n \geq 4$. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. A set of vertices is *independent* if all pairs of its vertices are non-adjacent. The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent vertex set in G. A *clique* is a complete subgraph maximal under inclusion (i.e., in our terminology different cliques in the same graph may have different size). The *clique number* $\omega(G)$ is the maximum number of vertices of a clique in G. The *clique covering number* $\theta(G)$ is the minimum cardinality of a set of cliques that covers all vertices of G. A graph G is *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G.

As usual, the *open neighborhood* $N(v)$ of v is the set of neighbors of v, whilst its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. In a less usual way, we also refer to the subgraphs induced by them as $N(v)$ and $N[v]$, respectively.

Throughout this paper, the notation $K_n - G$ will refer to the graph obtained from the complete graph K_n by deleting the *edge set* of a subgraph isomorphic to G. In this way, for instance, $K_4 - K_3$ means the claw $K_{1,3}$.

1.2. New definitions and terminology

In this paper, we use the following special terminology for some types of graphs.

- The *elementary types* are:
  (a) the wheel W_4,
  (b) the square C_n^2 of a cycle of length $n \geq 7$.
- The *supplementary types* are the following ones. (For illustration, see Fig. 1.)
  (A) $S_A = (V_A, E_A)$, where $V_A = \{v_i, u_i \mid 1 \leq i \leq 4\}$ and

$$E_A = \{v_i v_{i+1} \mid 1 \leq i \leq 4\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid 1 \leq i \leq 4\}$$

  (subscript addition taken modulo 4).
  (B) $S_B = (V_B, E_B)$, where $V_B = \{v_i \mid 1 \leq i \leq 5\} \cup \{u_1, u_2, u_3\}$ and

$$E_B = \{v_i v_{i+1} \mid 1 \leq i \leq 5\} \cup \{v_3 v_5, v_4 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid 1 \leq i \leq 3\}$$

  (subscript addition taken modulo 5).
  (C) $S_C = (V_C, E_C)$, where $V_C = \{v_i \mid 1 \leq i \leq 6\} \cup \{u_1, u_2\}$ and

$$E_C = \{v_i v_{i+1} \mid 1 \leq i \leq 6\} \cup \{v_2 v_4, v_3 v_5, v_4 v_6, v_5 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid i = 1, 2\}$$

  (subscript addition taken modulo 6).
  (D) $S_D = (V_D, E_D)$, where $V_D = \{v_i \mid 1 \leq i \leq 6\} \cup \{u_1, u_4\}$ and

$$E_D = \{v_i v_{i+1} \mid 1 \leq i \leq 6\} \cup \{v_1 v_3, v_2 v_4, v_4 v_6, v_5 v_1\} \cup \{u_i v_{i-1}, u_i v_i, u_i v_{i+1} \mid i = 1, 4\}$$

  (subscript addition taken modulo 6).

We also define two operations as follows.

- Suppose that $e = xy$ is an edge *contained in exactly one triangle* xyz, whilst xz and zy belong to more than one triangle. An *edge splitting* of e means replacing e with the 3-path xwy (where w is a new vertex) and inserting the further edge wz.
- Let u and v be two vertices *at distance at least* 4 *apart*. The *vertex sticking* of u and v means removing u and v and introducing a new vertex w adjacent to the entire $N(u) \cup N(v)$.[1]

The inverses of these operations can also be introduced in a natural way.

1 'Vertex sticking' and its inverse operation were also introduced in [11].