Discrete Applied Mathematics 209 (2016) 276–286

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Vertex-distinguishing proper arc colorings of digraphs[☆]

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ARTICLE INFO

Article history:
Received 17 August 2014
Received in revised form 31 August 2015
Accepted 27 October 2015
Available online 18 February 2016

Keywords:
Vertex-distinguishing arc colorings
Balanced arc colorings
Equitable arc colorings
Digraphs

ABSTRACT

An arc coloring of a digraph D is *proper* if (i) no two arcs with a common tail receive the same color and (ii) no two arcs with a common head receive the same color. Define the *out-arc set* and *in-arc set* of a vertex v of D to be the set of arcs with tail v and the set of arcs with head v, respectively. A proper arc coloring of D is *vertex-distinguishing* (resp. *semi-vertex-distinguishing*) if (i) no two vertices (resp. no three vertices) have the same color set for their in-going arcs and (ii) no two vertices (resp. no three vertices) have the same color set for their out-going arcs. And a proper arc coloring of D is *equitable* if the numbers of arcs colored by any two colors differ by at most one.

In this paper, (semi-)vertex-distinguishing proper arc colorings of digraphs are introduced. Denote by $\chi'_{vd}(D)$ (resp. $\chi'_{svd}(D)$) the minimum number of colors required for a vertex-distinguishing (resp. semi-vertex-distinguishing) proper arc coloring of D. We give upper bounds for $\chi'_{vd}(D)$ and $\chi'_{svd}(D)$ respectively. In particular, the value of $\chi'_{vd}(D)$ is obtained for some regular digraphs D. Moreover, we show that the values of $\chi'_{vd}(D)$ and $\chi'_{svd}(D)$ will not be changed if the coloring is, in addition, required to be equitable.

1. Introduction

Throughout the paper, all undirected graphs and digraphs considered are finite and, unless otherwise stated, simple, i.e., without parallel edges (arcs) or loops. We use Bang-Jensen and Gutin [5] for terminology and notation not defined here.

A proper edge coloring of an undirected graph is *vertex-distinguishing* if no two vertices are incident with the same set of colors. *Vertex-distinguishing proper edge colorings* (abbreviated *VDPE colorings*) of undirected graphs were introduced and studied independently by Aigner et al. [1], by Burris and Schelp [8] and by Horňk [12]. Note that an undirected graph has a VDPE coloring if and only if it contains no isolated edge and at most one isolated vertex. Such an undirected graph is referred to as a *vertex-distinguishing edge-colorable graph* (abbreviated *vdec-graph*).

For a vdec-graph G with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, let $n_d(G)$ be the number of vertices with degree d in G and let $\chi'_{vd}(G)$ be the minimum number of colors required for a VDPE coloring of G. Let

$$\pi(G) = \min\left\{k \in \mathbb{N} : \binom{k}{d} \geq n_d(G) \text{ for } \delta(G) \leq d \leq \Delta(G)\right\}. \quad (1)$$

In 1997, Burris and Schelp [8] conjectured that $\chi'_{vd}(G) \leq |V(G)| + 1$. Bazgan et al. [6] verified this conjecture in 1999.

☆ Supported by NSFC (No. 11171273 and No. 11301371), CSC (No. 2011629015) and NPU (No. G2015KY0301).
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http://dx.doi.org/10.1016/j.dam.2015.10.039
0166-218X/

Theorem 1.1 (*Bazgan et al. [6]*). *Let G be a vdec-graph on n vertices. Then* $\chi'_{vd}(G) \le n + 1$.

Burris and Schelp [8] also proposed the following conjecture.

Conjecture 1.1. *Let G be a vdec-graph. Then* $\chi'_{vd}(G) \in \{\pi(G), \pi(G) + 1\}$.

Conjecture 1.1 remains open and in fact even the weaker conjecture $\chi'_{vd}(G) \le \pi(G) + c$ for some fixed constant c cannot be proved till now. But some special graphs have been verified for Conjecture 1.1, including complete graphs, complete bipartite graphs, paths, cycles and some trees by Burris and Schelp [8], union of paths, union of cycles by Balister [3], two families of cubic graphs, ladders and unions of K_4, by Taczuk and Woźniak [14] and any graph G with $\Delta(G) \ge \sqrt{2|V(G)|} + 4$ and $\delta(G) \ge 5$ by Balister et al. [4]. Besides, Bazgan et al. [7] showed that $\chi'_{vd}(G) \le \Delta(G) + 5$ if $\delta(G) > |V(G)|/3$, Balister et al. [3] showed that $\pi(G) \le \chi'_{vd}(G) \le \pi(G) + 5$ if $\Delta(G) = 2$. For more details, we refer the readers to [2–4,7,9,13,14]. Recall the result of Burris and Schelp [8] for complete graphs K_n on $n \ge 3$ vertices

$$\chi'_{vd}(K_n) = \begin{cases} n, & n \text{ is odd,} \\ n + 1, & n \text{ is even.} \end{cases} \tag{2}$$

It follows that Conjecture 1.1 is best possible if it is true and the result in Theorem 1.1 cannot be improved in general.

Motivated by the problem for undirected graphs mentioned above, we introduce and study the analogous problem for digraphs, i.e., *vertex-distinguishing proper arc colorings* (abbreviated *VDPA colorings*) of digraphs. Some definitions and notations that will be used are given first.

Let $D = (V(D), A(D))$ be a simple digraph. An arc coloring of D is *proper* if (i) no two arcs with a common tail receive the same color and (ii) no two arcs with a common head receive the same color. Denote by $\chi'(D)$ the minimum number of colors required for a proper arc coloring of D.

Let v be a vertex of D. Define the out-arc set and in-arc set of v to be the set of arcs with tail v and the set of arcs with head v, respectively. In an arc coloring f of D, define the *out-arc color set* of v, $F^+(v)$, and *in-arc color set* of v, $F^-(v)$, to be the sets of colors of the arcs in its out-arc set and in-arc set, respectively. Note that both $F^+(v)$ and $F^-(v)$ could be empty sets.

Let $n_S^+ = n_S^+(D)$ (resp. $n_S^- = n_S^-(D)$) be the number of vertices with out-arc set (resp. in-arc set) assigned color set S. A proper arc coloring of D is called *vertex-distinguishing* (resp. *semi-vertex-distinguishing*) if $n_S^+ \le 1$ and $n_S^- \le 1$ (resp. $n_S^+ \le 2$ and $n_S^- \le 2$) for any color set S. A digraph is a *vdac*-digraph, vertex-distinguishing arc-colorable, if it has a VDPA coloring. Similarly, we can define a *semi-VDPA coloring* and a *svdac-digraph*. Clearly, every vdac-digraph is also a svdac-digraph.

Note that an isolated vertex can be regarded both as a source and as a sink. One can check that the following fact holds.

Fact 1.1. *A digraph D is a vdac-digraph (resp. svdac-digraph) if and only if D contains at most one source (resp. two sources) and at most one sink (resp. two sinks).*

For a vertex v of D, let $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$ be the outdegree and indegree of v, respectively. Denote by $\delta^+(D), \delta^-(D), \Delta^+(D)$ and $\Delta^-(D)$ the minimum outdegree, minimum indegree, maximum outdegree and maximum indegree of D, respectively. Let

$$\delta(D) = \min\{\delta^+(D),\ \delta^-(D)\}, \qquad \Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}. \tag{3}$$

For an integer d, let $n_d^+(D)$ and $n_d^-(D)$ be the numbers of vertices with outdegree d and indegree d in D, respectively. The minimum number of colors required for a VDPA coloring of a vdac-digraph D, the *vertex-distinguishing arc chromatic number*, is denoted by $\chi'_{vd}(D)$. Let

$$\pi(D) = \min\left\{ k \in \mathbb{N} : \begin{matrix} \binom{k}{d} \ge n_d^+(D) & \text{for } \delta^+(D) \le d \le \Delta^+(D) \\ \binom{k}{d} \ge n_d^-(D) & \text{for } \delta^-(D) \le d \le \Delta^-(D) \end{matrix} \right\}. \tag{4}$$

It is clear that $\chi'_{vd}(D) \ge \max\{\chi'(D), \pi(D)\}$. Note that $\chi'(D) = \Delta(D)$ (see Fact 2.1 in Section 2) and $\pi(D) \ge \Delta(D)$. Thus $\chi'_{vd}(D) \ge \pi(D)$. Analogous to Conjecture 1.1 for undirected graphs, we propose the following conjecture for digraphs.

Conjecture 1.2. *Let D be a vdac-digraph. Then* $\chi'_{vd}(D) = \pi(D)$.

Despite Conjecture 1.2 remains unsolved, some good progresses concerning it have been obtained. In particular, we get the following result.

Theorem 1.2. *Let D be a vdac-digraph on n vertices and* $t \ge 1$ *an integer. If* $\delta(D) \ge \frac{n-1}{t}$, *then* $\chi'_{vd}(D) \le \min\{n, \Delta(D) + t\}$.

Corollary 1.1. *Let D be a vdac-digraph on n vertices. Then* $\chi'_{vd}(D) \le n$.