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Hamiltonian cycles in spanning subgraphs of line graphs

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ABSTRACT

Let G be a graph and $e = uv$ an edge in G (also a vertex in the line graph $L(G)$ of G). Then e is in two cliques $E_G(u)$ and $E_G(v)$ with $E_G(u) \cap E_G(v) = \{e\}$ of $L(G)$, that correspond to all edges incident with u and v in G respectively. Let $SL(G)$ be any spanning subgraph of $L(G)$ such that every vertex $e = uv$ is adjacent to at least $min\{d_G(u) - 1, \lceil \frac{3}{4}d_G(u) + \frac{1}{2} \rceil\}$ vertices of $E_G(u)$ and to at least $min\{d_G(v) - 1, \lceil \frac{3}{4}d_G(v) + \frac{1}{2} \rceil\}$ vertices of $E_G(v)$. Then if $L(G)$ is Hamiltonian, we show that $SL(G)$ is Hamiltonian. As a corollary we obtain a lower bound on the number of edge-disjoint Hamiltonian cycles in $L(G)$.

1. Introduction

In this paper, a *graph* will be finite undirected graph $G = (V(G), E(G))$ without loops or multiple edges. For any vertex v of G, we denote the set of edges which are incident with v by $E_G(v)$ and the degree of v in G by $d_G(v)$. A *cycle* is a 2-regular connected subgraph of a graph. And a *Hamiltonian cycle* is a spanning cycle, i.e., the cycle visits each vertex of the graph exactly once. A graph is called *Hamiltonian* if it has a Hamiltonian cycle. A subgraph D in G is called *dominating* if each edge of G is either in D or incident to a vertex in D.

The *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$, where two vertices of $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges in G are incident with a common vertex in G. The complete bipartite graph $K_{1,3}$ is called a *claw*. We say a graph G is *claw-free* if G contains no claw as an induced subgraph. For terminology and notation not defined here we refer to [2].

The Hamiltonicity of line graphs has been widely studied by various researchers since Harary and Nash-Williams [3] obtained the following theorem in 1965.

Theorem 1 (*Harary and Nash-Williams [3]*)**.** *Let G be a graph not a star. Then $L(G)$ is Hamiltonian if and only if G has a dominating closed trail.*

The following two well-known conjectures proposed by Thomassen [9] and Matthews and Sumner [7] respectively have been standed among the most interesting unsolved problems in Hamiltonian graph theory.

Conjecture 1 (*Thomassen [9]*)**.** *Every 4-connected line graph is Hamiltonian.*

and

Conjecture 2 (*Matthews and Sumner [7]*)**.** *Every 4-connected claw-free graph is Hamiltonian.*

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Clearly every line graph is claw-free, Conjecture 1 is a special case of Conjecture 2.

Ryjáček [8] introduced a powerful tool, which is called *line graph closure*. He showed that for every claw-free graph G there is a graph $cl(G)$, which is called a closure of G, such that G is a spanning subgraph of $cl(G)$, $cl(G)$ is the line graph of a triangle-free graph and $cl(G)$ maintains the circumference of G. With this argument, he proved the two conjectures above are equivalent.

Theorem 2 (*Ryjáček [8]*). *Conjectures 1* and *2 are equivalent.*

Two important results about Conjectures 1 and 2 were obtained. For Conjecture 1, the first nice result was due to Zhan [11], as follows.

Theorem 3 (*Zhan [11]*). *Every 7-connected line graph is Hamiltonian.*

Kaiser and Vrána improved Theorem 3 for 5-connected line graphs.

Theorem 4 (*Kaiser and Vrána [4]*). *Every 5-connected claw-free graphs with minimum degree at least 6 is Hamiltonian.*

Another approach towards Conjectures 1 and 2 was first done by Lai *et al*. They considered the essential connectivity in the following result, where a graph G is essentially k-connected if G does not have a vertex cut X with $|X| < k$ such that $G - X$ has at least two components of at least two vertices.

Theorem 5 (*Lai et al. [5]*). *Every 3-connected, essentially 11-connected line graph is Hamiltonian.*

Yang *et al*. and Li and Yang strengthened Theorem 5 to Hamiltonian connected property and to hamiltonicity of essentially 10-connected line graphs respectively.

Theorem 6 (*Yang et al. [10]*). *Every 3-connected essentially 11-connected line graph (claw-free graph) is Hamilton-connected.*

Theorem 7 (*H. Li and W. Yang [6]*). *Every 3-connected essentially 10-connected line graph (claw-free graph) is Hamilton-connected.*

Clearly from the definition of line graph, for every vertex v in G, the vertices that correspond to all edges in $E_G(v)$ form a clique, for convenience which is also denoted by $E_G(v)$ in the line graph $L(G)$ of G. If G has a dominating closed trail, along this trail (in a chosen direction) we consider every such clique in $L(G)$, then we may construct directly a Hamiltonian cycle in $L(G)$. A clique has so many edges that we can choose paths between a pair of vertices easily in this clique. In other words, the line graph has so many edges to guarantee the existence of a Hamiltonian cycle.

In fact the line graph closure method of Ryjáček is to add edges to the claw-free graph and to guarantee that the resulting graph is a line graph. And in this closure operation, the Hamiltonian property is always kept. We are interested in an inverse operation: if some edges of a Hamiltonian line graph are deleted, then can the resulting spanning subgraph still be Hamiltonian?

These lead us to the following problem: under which conditions a spanning subgraph of a Hamiltonian line graph is Hamiltonian. In this paper we focus to study the problem on spanning subgraphs that can be obtained from $L(G)$ by deleting no more than $max\{0, \lfloor \frac{1}{4}|E_G(v)| - \frac{1}{2} \rfloor - 1\}$ edges in the clique $E_G(v)$ incident with every vertex $v \in V(G)$.

First we define those spanning subgraphs of line graph which will be considered in this paper.

Definition. Let $G = (V(G), E(G))$ be a graph. A graph is called $SL(G)$ if

(1) this graph is a spanning subgraph of $L(G)$, and
(2) in this graph every vertex $e = uv$ (u and v are two end vertices of the edge e in G) is adjacent to at least $min\{d_G(u) - 1, \lceil \frac{3}{4}d_G(u) + \frac{1}{2} \rceil\}$ vertices of $E_G(u)$ and also adjacent to at least $min\{d_G(v) - 1, \lceil \frac{3}{4}d_G(v) + \frac{1}{2} \rceil\}$ vertices of $E_G(v)$.

Denote by $\mathcal{SL}(G)$ the family of all $SL(G)$. Obviously all $SL(G)$ can be obtained from $L(G)$ by deleting edges and $L(G) \in \mathcal{SL}(G)$. For any positive integer d, by simple calculation, we have the equality $d = \lceil \frac{3}{4}d + \frac{1}{2} \rceil + \lfloor \frac{1}{4}d - \frac{1}{2} \rfloor$ and hence for any vertex $v \in V(G)$, $d(v) - 1 - \lceil \frac{3}{4}d_G(v) + \frac{1}{2} \rceil = \lfloor \frac{1}{4}d(v) - \frac{1}{2} \rfloor - 1$. From $L(G)$ we delete edges in every clique $E_G(u)$ such that for any vertex $e = uv$, no more than $max\{0, \lfloor \frac{d_G(u)}{4} - \frac{1}{2} \rfloor - 1\}$ edges adjacent to $e = uv$ in the clique $E_G(u)$ are deleted. Then the resulting spanning subgraph is in $\mathcal{SL}(G)$. In particular, if the maximum degree of G is not more than 9, then $SL(G) = L(G)$.

The main result we obtain in this paper is about hamiltonicity of $SL(G)$.

Theorem 8. *For a graph G, if $L(G)$ is Hamiltonian, then $SL(G)$ is also Hamiltonian.*

By Theorems 1, 4 and 8, we can easily get the following two corollaries.

Corollary 9. *Let G be a graph not a star. Then $SL(G)$ is Hamiltonian if and only if G has a dominating closed trail.*

Corollary 10. *Every 5-connected $SL(G)$ with minimum degree at least 6 is Hamiltonian.*