Discrete Applied Mathematics 209 (2016) 296–302

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Minor relation for quadrangulations on the projective plane

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ARTICLE INFO

Article history:
Received 31 October 2014
Accepted 31 July 2015
Available online 1 September 2015

Keywords:
Contraction
Minor
Planar
Quadrangulation
Reduction
Projective plane

ABSTRACT

A *quadrangulation* on a surface is a map of a simple graph on the surface with each face quadrilateral. In this paper, we prove that for any bipartite quadrangulation G on the projective plane, there exists a sequence of bipartite quadrangulations on the projective plane $G = G_1, G_2, \dots, G_n$ such that

(i) G_{i+1} is a minor of G_i with $|G_i| - 2 \leq |G_{i+1}| \leq |G_i| - 1$, for $i = 1, \dots, n-1$,
(ii) G_n is isomorphic to either $K_{3,4}$ or $K_{4,4}^{--}$,

where $K_{4,4}^{--}$ is the graph obtained from $K_{4,4}$ by deleting two independent edges. In order to prove the theorem, we use two local reductions for quadrangulations which transform a quadrangulation Q into another quadrangulation Q' with $Q \geq_m Q'$ and $1 \leq |Q| - |Q'| \leq 2$. Moreover, we prove a similar result for non-bipartite quadrangulations on the projective plane.

1. Introduction

A *map* G on a surface F^2 is a fixed embedding of a simple graph on F^2. A *face* of G is a component of $F^2 - G$, and we suppose that each face is homeomorphic to an open 2-cell. A *region* of G is the union of several faces of G, and the *link* of a vertex v in G is the boundary walk of the 2-cell region of all faces incident to v in G. A *k-vertex* is a vertex of degree k and a k-cycle is a cycle of length k. For a cycle or path C in G, a *chord* of C is an edge $e = xy$ of G with $x, y \in V(C)$ and $e \notin E(C)$. A closed curve ℓ on F^2 is *contractible* if ℓ bounds a 2-cell on F^2, and ℓ is *essential* otherwise. We apply this definition to closed walks of maps by regarding them as closed curves on the surface. When we use symbols with subscripts to express vertices and edges, we take a suitable modulus.

A *triangulation* on a surface is a map on the surface such that each face is bounded by a 3-cycle. It is known that every triangulation on the sphere is reduced to the tetrahedron by edge contractions [17]. Barnette proved that every triangulation on the projective plane is reduced to either K_6 or $K_4 + \overline{K_3}$ by edge contractions [1]. There are similar results for triangulations on the torus [10] and the Klein bottle [11,18]. See [3,9] for related results.

In this paper, we consider reductions for quadrangulations on surfaces. A *quadrangulation* on a surface is a map on the surface such that each face is bounded by a 4-cycle. It is known that every quadrangulation on the sphere is bipartite, but this does not hold for any other surface.

Let G be a quadrangulation on a surface and let f be a face bounded by a 4-cycle $abcd$. Define a *face contraction* of f at $\{b, d\}$ to identify b and d and replace two pairs of multiple edges $\{ab, ad\}$ and $\{cb, cd\}$ with two single edges, respectively, as shown in Fig. 1. We do not apply this operation if the resulting graph is not simple. It is easy to see that a face contraction preserves the bipartiteness of G.

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http://dx.doi.org/10.1016/j.dam.2015.07.037
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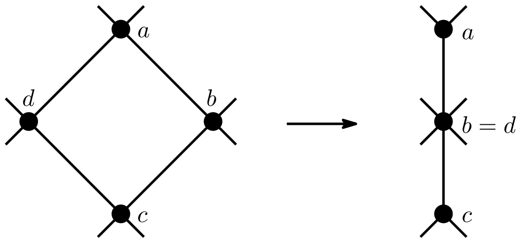

Fig. 1. A face contraction.

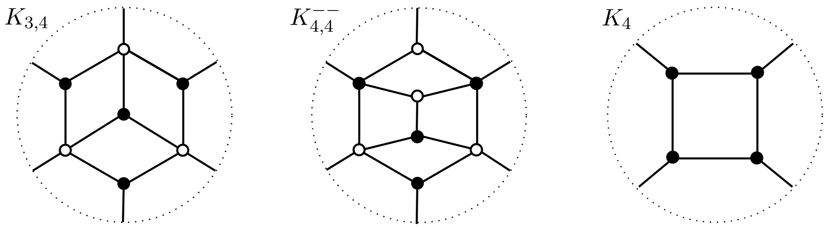

Fig. 2. $K_{3,4}$, $K_{4,4}^{--}$ and K_4.

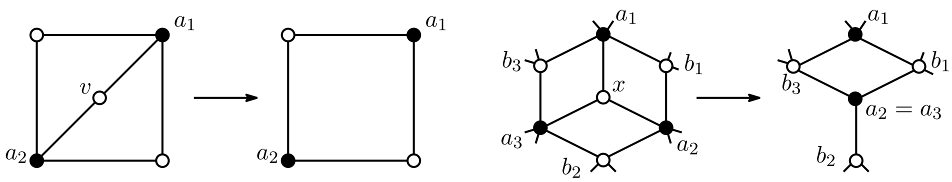

Fig. 3. A 2-vertex deletion and a hexagonal contraction.

Negami and Nakamoto [16] proved that every quadrangulation on the sphere can be reduced to a 4-cycle by face contractions. For the projective plane, they proved the following, where we let $\mathbb{P}$ denote the projective plane throughout the paper.

Theorem 1 (*Negami and Nakamoto [16]*)**.** *Every bipartite quadrangulation on the projective plane $\mathbb{P}$ can be reduced to $K_{3,4}$ by face contractions, through bipartite ones. Every non-bipartite quadrangulation on $\mathbb{P}$ can be reduced to K_4 by face contractions, through non-bipartite ones. (The two quadrangulations $K_{3,4}$ and K_4 on $\mathbb{P}$ are shown in Fig. 2.)*

For quadrangulations on the torus [14] and the Klein bottle [13], similar results are known. There are some related results for quadrangulations keeping minimum degree and some other conditions [4,8,15].

A graph H is a *minor* of another graph G, denoted by $H \leq_m G$, if H is obtained from G by contracting and deleting edges, where these two operations are called *minor operations*. (Let G/e and $G-e$ denote the graphs obtained from G by contracting and deleting an edge e, respectively.) Observe that a single edge contraction or a single edge deletion in a quadrangulation *never* transforms it into a quadrangulation. On the other hand, in Theorem 1, if a quadrangulation G can be reduced to G' by a single face contraction, then G' is not necessarily a minor of G [2]. So it seems to be difficult to combine reductions for quadrangulations with minor relations.

In [2], Bau et al. have defined two local reductions which transform a quadrangulation G into a quadrangulation G' with $G' \leq_m G$, as follows: a *2-vertex deletion* of a 2-vertex v is to delete v from G, as shown in the left hand of Fig. 3. So if we let G' be the resulting graph, then $|V(G')| = |V(G)| - 1$ and $G' = (G - va_1)/va_2$, where a_1 and a_2 are the two neighbors of v. A *hexagonal contraction* of a 3-vertex x at $\{a_2, a_3\}$ is to delete x, identify a_2 and a_3, and replace a pair of multiple edges $\{a_2b_2, a_3b_2\}$ with a single edge, as shown in the right hand of Fig. 3, where $a_1b_1a_2b_2a_3b_3$ is the link of x in G and a_1, a_2, a_3 are the neighbors of x in G. So if we let G' be the resulting graph, then $|V(G')| = |V(G)| - 2$ and $G' = (((G - xa_1)/xa_2)/xa_3) - a_2b_2$.

We do not apply these operations if the resulting graph is not simple. We see that these operations preserve the bipartiteness of quadrangulations, as well as the face contractions, and are obtained as a combination of contractions and deletions of edges.

Theorem 2 (*Bau et al. [2]*)**.** *Every quadrangulation on the sphere can be reduced to a* 4*-cycle by* 2*-vertex deletions and hexagonal contractions, only through quadrangulations.*

In this paper, we establish an analogy of Theorem 2 for quadrangulations on the projective plane, using 2-vertex deletions and hexagonal contractions, and prove the following theorem.