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Note

A simple branching scheme for vertex coloring problems

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ABSTRACT

We present a branching scheme for some vertex coloring problems based on a new graph operator called *extension*. The extension operator is used to generalize the branching scheme proposed by Zykov for the basic problem to a broad class of coloring problems, such as graph multicoloring, where each vertex requires a multiplicity of colors, graph bandwidth coloring, where the colors assigned to adjacent vertices must differ by at least a given distance, and graph bandwidth multicoloring, which generalizes both the multicoloring and the bandwidth coloring problems. We report some computational evidence of the effectiveness of the new branching scheme.

1. Introduction

Given a graph $G = (V, E)$ and an integer k, a k-coloring of the graph G is a mapping of each vertex to a single color such that adjacent vertices take different colors. The minimum graph coloring problem (Min–GCP), which is known to be NP-hard, consists in finding the minimum k such that a k-coloring exists. Such a minimum k is the chromatic number of G and is denoted by $\chi(G)$, or simply by χ. A second graph coloring problem that generalizes Min–GCP asks us to assign to every vertex a given number of colors, while satisfying the constraint that the same color cannot appear at adjacent vertices. This problem is known in the literature as the minimum graph multicoloring problem (Min–GMP), and though it can be reduced to Min–GCP on an auxiliary (much bigger) graph, it has its own interest and ad hoc algorithms can take advantage of its structure. Let $\chi_m(G)$ denote the minimum k for Min–GMP. A third version of the vertex coloring problem, defined on an edge-weighted graph, consists of assigning to each vertex a set of colors such that the distance between any color assigned to every pair of adjacent vertices is at least equal to the edge weight. This version of the problem is known in the literature as the bandwidth multicoloring problem, or bandwidth coloring if every vertex requires a single color. Let $\chi_{mb}(G)$ and $\chi_b(G)$ denote the minimum number of colors k for Min–GBMP and for the simple bandwidth coloring problem, respectively. For a recent survey on vertex graph coloring problems, see [6].

Branching schemes in implicit enumeration approaches such as Branch&Bound, Branch&Cut and Branch&Price are of utmost importance for the efficiency of the algorithms, especially when strongly structured problems are at hand. The addition of general branching constraints often risks spoiling the structure of the problem. Zykov in 1949 proposed a simple bipartite branching scheme for Min–GCP whose main feature is the generating in both branches of two smaller instances of the same problem. For the other coloring problems, such as Min–GMP, Min–GBMP and the bandwidth coloring problem, to the best of our knowledge, no branching schemes with this characteristic have been proposed.

In the Branch&Price algorithm proposed in [8] for Min–GMP, the branching schemes adopted are quite involved and mainly fix variables corresponding to columns of the extended formulation to 0 or to 1. In the case of frequency assignment

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problems (e.g., [1]), which basically are Min–GBMP problems on graphs with a special structure, the branching is carried out on binary variables that indicate the assignment of colors (frequencies) to vertices (cells). More involved branching schemes are also devised [3,5], but they are always derived from the mathematical programming formulation of the problem.

In the following section, we introduce a new operator generalizing the basic Zykov branching scheme that exploits the problem structure, and keeps the graph size small. Some computational evidence of the effectiveness of the branching scheme is reported.

2. The extension operator and the branching scheme

Given a graph G and two non-adjacent vertices v and w, consider the following two operations: *contraction* and *insertion*. The contraction (or shrinking) operation, denoted by $G \setminus \{v, w\}$, consists of creating a new vertex u with an incident edge for each vertex in the union of the neighborhoods of v and w and of removing vertices v and w from the vertex set. The insertion operation consists of adding the edge $\{v, w\}$ to E, and is denoted by $G + \{v, w\}$. As a corollary to a theorem in [10], we have

Theorem 1. $\chi(G) = \min\{\chi(G \setminus \{v, w\}), \chi(G + \{v, w\})\}$ [10].

This well-known result has been exploited in a number of coloring algorithms that are usually called Zykov's (contraction) algorithms (see, e.g., [2,7]). The branching scheme derived from this result considers two non-adjacent vertices of G, v and w. The alternatives are: either v and w are assigned to different colors, enforced by the introduction of edge $\{v, w\}$ $(G+\{v, w\})$, or they are assigned to the same color, as a result of the contraction of vertices v and w $(G \setminus \{v, w\})$. In this case, note that the color of vertex u, corresponding to the color assigned to both v and w, must be different from the colors of vertices previously adjacent to v or to w. One of the best exact approaches to graph coloring is a Branch&Price algorithm, as pioneered in [9] and recently revived in e.g. [4], that makes use of a branching rule equivalent to the one derived from Theorem 1.

Let us consider now Min–GMP and Min–GBMP, which are formally stated as follows.

Min–GMP. Given a graph $G = (V, E)$ and integer weights $c_v > 0$ on the vertices $v \in V$, assign c_v colors to each vertex v such that the colors assigned to every pair of adjacent vertices v and w (i.e. $\{v, w\} \in E$) are all different. The objective is to minimize the total number of colors used.

Min–GBMP. Given a graph $G = (V, E)$ with integer weights $c_v > 0$ on the vertices $v \in V$ and integer weights $b_{vw} > 0$ on the edges, assign c_v colors to each vertex v such that the colors assigned to every pair of adjacent vertices v and w (i.e. $\{v, w\} \in E$) are all different; moreover, identifying colors with natural numbers, the difference between any color assigned to v and any color assigned to w must be at least b_{vw} for every $\{v, w\} \in E$. The objective is to minimize the bandwidth of the colors used, that is the difference between the highest and the lowest color used. In other cases the objective can be the minimization of the number of colors used.

We introduce now an operator that generalizes Zykov contraction and insertion described above. Consider two non-adjacent vertices v and w of G, and let $N(v)$ be the neighborhood of v. Let $\uplus_t$ denote the new operator called *extension*, where t is an integer parameter having value between 0 and $\min\{c_v, c_w\}$.

Definition 1. $G' = (V', E') = G \uplus_t \{v, w\}$ is obtained as follows:

- $V' = V \cup \{u\}$, $c_u = t$.
- $c_v \leftarrow c_v - t$ and $c_w \leftarrow c_w - t$;
- E' includes the following new edges, in addition to those of E:
  - $\{v, w\}$, $\{v, u\}$, and $\{w, u\}$ with weights $b_{vw} = b_{vu} = b_{wu} = 1$;
  - $\{u, u'\}$ with weight $b_{uu'} = b_{vu'}$, for each vertex $u' \in N(v) \setminus N(w)$;
  - $\{u, u'\}$ with weight $b_{uu'} = b_{wu'}$, for each vertex $u' \in N(w) \setminus N(v)$;
  - $\{u, u'\}$ with weight $b_{uu'} = \max\{b_{vu'}, b_{wu'}\}$, for each vertex $u' \in N(v) \cap N(w)$;
- finally all vertices s with $c_s = 0$ and all their incident edges are removed from G'.

An application of the extension operator is exemplified in Fig. 1. Note that the extension operator generalizes Zykov insertion and contraction. Consider the classical vertex coloring problem, equivalent to Min–GBMP where all weights c_v and b_{vw} are equal to 1. In this case t is either 0 or 1: when $t = 0$, the operation $G \uplus_0 \{v, w\}$ is equivalent to the insertion of edge $\{v, w\}$, since the new vertex u has weight $c_u = 0$ and is omitted. When $t = 1$, the operation $G \uplus_1 \{v, w\}$ is equivalent to the contraction of v and w, since $c_v = c_w = 0$ and the vertices are removed.

By analogy with the Zykov result, we can prove that the extension operator can be used within a branching scheme to decrease the complexity of the graph coloring instance, and eventually reduce to easy subproblems.

Theorem 2. *Consider an instance of* Min–GBMP *defined on graph G, and two non-adjacent vertices v and w, and let $h = \min\{c_v, c_w\}$ be the maximum number of colors that v and w can share; then*

$$\chi_{mb}(G) = \min_{t=0\cdots h} \chi_{mb}(G \uplus_t \{v, w\}).$$