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Adjacency polynomials of digraph transformations[☆]

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ABSTRACT

Let $A(\lambda, D)$ be the adjacency characteristic polynomial of a digraph D. In the paper Deng and Kelmans (2013) the so-called (xyz)-transformation D^{xyz} of a simple digraph D was considered, where $x, y, z \in \{0, 1, +, -\}$, and the formulas of $A(\lambda, D^{xyz})$ were obtained for every r-regular digraph D in terms of r, the number of vertices of D, and $A(\lambda, D)$. In this paper we define the so-called $(xyab)$-transformation D^{xyab} of a simple digraph D, where $x, y, a, b \in \{0, 1, +, -\}$. This notion generalizes the previous notion of the (xyz)-transformation D^{xyz}, namely, $D^{xyab} = D^{xyz}$ if and only if $a = b = z$. We extend our previous results on $A(\lambda, D^{xyz})$ to the $(xyab)$-transformation D^{xyab} by obtaining the formulas of $A(\lambda, D^{xyab})$, where $x, y, a, b \in \{0, 1, +, -\}$ and $a \neq b$, for every simple r-regular digraph D in terms of r, the number of vertices of D, and $A(\lambda, D)$. We also use $(xyab)$-transformations to describe various constructions providing infinitely many examples of adjacency cospectral non-isomorphic digraphs.

1. Introduction

We will consider directed graphs which we call *digraphs* for short. All notions on graphs and matrices that are used but not defined here can be found in [3,14,15,17].

Various important results in graph theory were obtained by introducing some operations on graphs and establishing how these operations affect certain properties or parameters of graphs. For example, the Bondy–Chvatal and Ryzáček closers of graphs are very useful operations in graph Hamiltonicity theory [3]. (Some strengthenings and extensions of the Ryzáček result are given in [18].) Kelmans' graph transformations (see, for example, [4,19]) turned out to be very useful because they are monotone with respect to various partial order relations on the set of graphs. Gross and Tucker introduced the operation of voltage lifting on a graph which can be generalized to digraphs [11,16]. By this operation one can obtain the so-called derived covering (di)graph and the relation between the adjacency characteristic polynomials of the (di)graph and its derived covering (di)graph [11,10,24].

In this paper we consider certain operations on digraphs depending on parameters $x, y, a, b \in \{0, 1, +, -\}$. These operations induce functions $T^{xyab} : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a class of digraphs of certain type and $x, y, a, b \in \{0, 1, +, -\}$. The class $\mathcal{S}$ of digraphs we consider is mainly the class of simple digraphs $\mathcal{D}$. We will also mention some results on digraphs with loops and/or parallel arcs. For $D \in \mathcal{D}$, we put $T^{xyab}(D) = D^{xyab}$ and call D^{xyab} the *$(xyab)$-transformation* of D. When $a = b = z$, the $(xyab)$-transformation of D is just the (xyz)-transformation of D introduced and studied in [7].

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For a digraph D the adjacency characteristic polynomials of D^{00+}, D^{+0+}, D^{0++}, and D^{+++} were described by Zhang, Lin, and Meng [29]. The adjacency characteristic polynomials of the other transformations D^{xyz} of a regular digraph D with $x, y, z \in \{+, -\}$ were obtained by Liu and Meng [22]. Deng and Kelmans presented in [7] the adjacency characteristic polynomials of all (xyz)-transformations of a regular digraph D and discussed the relations between those transformations. They also obtained in [8] the Laplacian characteristic polynomials of all (xyz)-transformations of a regular digraph D and some non-regular digraphs. The Laplacian spectra of $(xyab)$-transformations of a regular digraph D, when $a \neq b$, were described in [6].

For an undirected graph G, some graph properties of the transformation graphs G^{xyz} with $x, y, z \in \{+, -\}$ were discussed and obtained in [21,27,26]. For a regular undirected graph G, the adjacency characteristic polynomial and spectrum of G^{00+}, G^{+0+}, G^{0++} and G^{+++} were given in [5] (pages 63 and 64). Yan and Xu obtained the adjacency spectra of the other seven transformations G^{xyz} with $x, y, z \in \{+, -\}$ in terms of the adjacency spectrum of G [28]. In 1967 Kelmans established the formulas for the Laplacian characteristic polynomials and the number of spanning trees of G^{0++}, G^{0+0}, G^{00+} and of the line graph of G [20]. In [9] Deng, Kelmans, and Meng presented the formulas for the Laplacian polynomials and the number of spanning trees of G^{xyz} in terms of the Laplacian characteristic polynomial of G for all $x, y, z \in \{0, 1, +, -\}$. In [23] Li and Zhou described the signless Laplacian characteristic polynomials for G^{xyz} for a simple regular graph G and $x, y, z \in \{0, 1, +, -\}$ in terms of the signless Laplacian characteristic polynomial of G.

In this paper we study the adjacency spectra of $(xyab)$-transformations D^{xyab} for a regular digraph D, where $x, y, a, b \in \{0, 1, +, -\}$. Our main goal is to describe the formulas for the adjacency characteristic polynomials of D^{xyab} for an r-regular digraph D in terms of r, the adjacency characteristic polynomial of D, and the number of vertices of D. We also establish isomorphisms between some transformations and their inverses. Furthermore, we describe some adjacency cospectral but non-isomorphic $(xyab)$-transformations.

In Section 2 we introduce main notions and notation.

Section 3 contains some preliminaries.

In Section 4 we describe the adjacency characteristic polynomials of transformations D^{xyab} with $x, y, a, b \in \{0, 1, +, -\}$ and $a \neq b$.

In Section 5 we establish some conditions guaranteeing that two different $(xyab)$-transformations of a given digraph are isomorphic. In this section we also present some results on adjacency cospectral but non-isomorphic transformations and some examples. In addition, we introduce the notion of the core of a digraph and illustrate its usefulness in the study of the adjacency characteristic polynomials of digraphs.

In the Appendix we provide for all $x, y, a, b \in \{0, 1, +, -\}$ the list of formulas for the adjacency characteristic polynomials of the $(xyab)$-transformations of an r-regular digraph D in terms of the adjacency characteristic polynomial $A(\lambda, D)$ and the numbers r, n and $m = nr$, where n and m are the numbers of vertices and arcs of D, respectively.

2. Some notions and notation

A *digraph* D is a pair (V, E), where $V = V(D)$ is a finite non-empty set and $E = E(D) \subseteq V \times V$. The elements of V and D are called *vertices* and *arcs* of D, respectively. By this definition, D has no "parallel arcs" (i.e. no two different arcs a and b having the same vertex pair (u, v)). As we will indicate later, some of our results below will be also true for digraphs with "parallel" arcs as well.

Given two digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$, a bijection $\alpha : V_1 \to V_2$ is called an *isomorphism from* D_1 *to* D_2 if $(u, v) \in E_1 \Leftrightarrow (\alpha(u), \alpha(v)) \in E_2$. If there exists an isomorphism from D_1 to D_2 we say that D_1 *is isomorphic to* D_2, or equivalently, D_1 *and* D_2 *are isomorphic*.

An arc e of D is called a *loop* if $e = (v, v)$ for some $v \in V$. Obviously, D may have at most one loop in every vertex. A digraph $D = (V, E)$ is called *simple* if D has no loops. Let $\mathcal{D}$ denote the set of simple digraphs.

Let $v(D) = |V(D)|$ and $e(D) = |E(D)|$. For a digraph $D = (V, E)$ and $X \subseteq E$, let $D \setminus X = (V, E \setminus X)$. Obviously, $D \setminus X$ is also a digraph.

Let $K(V)$ denote the digraph (V, A), where $A = V \times V \setminus \{(v, v) : v \in V\}$. Graph $K(V)$ is called *the simple complete digraph with the vertex set* V. For a simple digraph $D = (V, E)$, let $D^c = K(V) \setminus E$. We say that digraph D^c is *simple complement to* D (or D and D^c are *simple complement*). We call $K^c(V) = (V, \emptyset)$ the *arc-empty digraph* with the vertex set V.

A digraph D is *connected* (*not connected*) if the undirected graph obtained from D by ignoring the orientations of the arcs in D is connected (resp., not connected).

A *component* of a digraph D is a maximal connected subgraph of D. Obviously, two different components of D are disjoint (i.e. have no common vertex).

If $e \in E$ is an arc in D and $e = (u, v)$ (possibly, $u = v$), then u is called the *tail of the arc* e and v is called the *head of the arc* e; we put $t(e, D) = t(e) = u$ and $h(e, D) = h(e) = v$. Obviously, t and h are functions: $t : E \to V$ and $h : E \to V$. The functions t and h can also be described by the corresponding $(V \times E)$-matrices $T(D)$ and $H(D)$ (see, for example, [25]):

(t) the *tail incidence matrix* $T(D) = (t_{ij})$ of D, where $t_{ij} = 1$ if $v_i = t(e_j)$ and $t_{ij} = 0$, otherwise, and

(h) the *head incidence matrix* $H(D) = (h_{ij})$ of D, where $h_{ij} = 1$ if $v_i = h(e_j)$ and $h_{ij} = 0$, otherwise.

The *line digraph* of D, denoted by D^l, is the digraph with vertex set $V(D^l) = E(D)$ and arc set $E(D^l) = \{(p, q) : p, q \in E(D) \text{ and } h(p, D) = t(q, D)\}$. A digraph D is called a *line digraph* (a *non-line digraph*) if there exists a digraph F (respectively, there is no digraph F) such that $D = F^l$.