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Dynamic coloring of graphs having no K_5 minor

Younjin Kim [a], Sang June Lee [b,*], Sang-il Oum [c,d]

[a] Department of Mathematics, Ewha Womans University, Seoul, South Korea
[b] Department of Mathematics, Duksung Women's University, Seoul, South Korea
[c] Department of Mathematical Sciences, KAIST, Daejeon, South Korea
[d] School of Mathematics, KIAS, Seoul, South Korea

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ABSTRACT

We prove that every simple connected graph with no K_5 minor admits a proper 4-coloring such that the neighborhood of each vertex v having more than one neighbor is not monochromatic, unless the graph is isomorphic to the cycle of length 5. This generalizes the result on planar graphs by S.-J. Kim, W.-J. Park and the second author [Discrete Appl. Math. 161 (2013) 2207–2212].

1. Introduction

In this paper, all graphs are assumed to be simple, meaning that they have no loops and no parallel edges. We say that a vertex v of a graph G is *dynamic* with a proper coloring of G, if either v has a pair of neighbors having distinct colors or v has at most one neighbor. A *dynamic k-coloring* of a graph G is a proper (vertex) k-coloring of G such that every vertex is dynamic. The *dynamic chromatic number* $\chi_d(G)$ of a graph G is the minimum number k such that G has a dynamic k-coloring.

This concept was introduced by Montgomery [25]. Dynamic chromatic numbers (and list dynamic chromatic numbers) have been studied for various classes of graphs such as graphs of small maximum degree, bipartite graphs, regular graphs, random graphs, and graphs embedded in a surface [25,22,21,24,1,23,2,16,5,13,15].

Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [7] initiated a similar but opposite concept called a local ℓ-coloring. A *local ℓ-coloring* is a proper coloring such that the neighbors of each vertex receive at most l colors. There are series of results in this concept as well [19,28,29,8,26,31,17,10,3,30].

Clearly, for every graph G, the number $\chi_d(G)$ is at least the chromatic number $\chi(G)$. It is easy to check that $\chi_d(C_5) = 5$ and $\chi(C_5) = 3$, and hence, $\chi_d(G)$ may be strictly larger than $\chi(G)$. Moreover, a graph with small chromatic number may have arbitrarily large dynamic chromatic number; for instance, if G is the graph obtained from K_n by subdividing every edge, then $\chi_d(G) = n$ but $\chi(G) = 2$. This might suggest that the dynamic chromatic number of a graph may be quite different from the usual chromatic number. However, it turns out that every connected planar graph except for C_5 has a dynamic 4-coloring, if we assume the four color theorem.

Theorem 1 (*S.-J. Kim, S. J. Lee, W.-J. Park [15]*)**.** *If G is a connected planar graph other than C_5, then G is dynamically 4-colorable.*

* Corresponding author.
E-mail addresses: younjinkim@ewha.ac.kr (Y. Kim), sjlee242@gmail.com, sanglee242@duksung.ac.kr (S.J. Lee), sangil@kaist.edu (S.-i. Oum).

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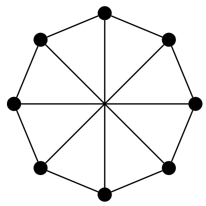

Fig. 1. The graph V_8.

In this paper, we consider the dynamic chromatic numbers of graphs with no K_5 minor. Note that Wagner [34] proved that such graphs are 4-colorable, assuming the four color theorem. Our main theorem is as follows.

Theorem 2. *If G is a connected graph other than C_5 having no K_5 minor, then G is dynamically 4-colorable.*

Our proof is based on the result in [15] and the structural characterization of graphs with no K_5 minor by Wagner [34]. Unlike the usual graph coloring, combining coloring on both side is not easy, because we should make sure that every vertex is dynamic.

Next, we consider graphs with no K_t minor for a general t and show the following.

Theorem 3. *For every integer $t \geq 2$, the following hold:*

(i) *A graph with no K_t topological minor is dynamically $(10t^2 + 2)$-colorable.*
(ii) *A graph with no K_t minor is dynamically $(\lfloor 64t\sqrt{\log_2 t} \rfloor + 3)$-colorable.*

See Section 5 for the proof of Theorem 3 and the related discussion.

Organization: In Section 2, we prove Theorem 2. In the proof, we will use two properties of minimum counterexamples, Lemmas 5 and 6, that are proved in Sections 3 and 4, respectively. In Section 5, we discuss a related question motivated by Hadwiger's conjecture and prove Theorem 3.

Notation: Let G be a graph. Let $V(G)$ denote the set of vertices of G. Let $E(G)$ denote the set of edges of G, and let $e(G) = |E(G)|$. For $u, v \in V(G)$, let $u \sim v$ denote that u is adjacent to v. Let $N_G(v)$ denote the set of neighbors of v in G. Let $\deg_G(v)$ denote the degree of v in G. For a vertex v and a vertex set U, let $e(v, U)$ denote the number of edges between v and vertices of U, that is, $e(v, U) = |N_G(v) \cap U|$. For $x, y \in V(G)$, let $G + xy$ be the graph obtained from G by adding the edge xy, and let G/xy be the graph obtained from G by identifying x and y.

2. Proof of Theorem 2

In order to prove Theorem 2, we will suppose that there is a counterexample of Theorem 2, and then, imply a contradiction. To this end, we show several properties of minimum counterexamples of Theorem 2. For the first property, we use the following definition.

Definition 4. A graph G with $|V(G)| > 3$ is called *internally* 3-*connected* if the following hold:

(a) G is 2-connected.
(b) For every separation (A, B) of order 2, we have that $|A \setminus B| = 1$ or $|B \setminus A| = 1$.

Lemma 5. *If G is a counterexample of Theorem 2 with minimum number of edges, then G is internally 3-connected.*

Our proof of Lemma 5 is given in Section 3. The following lemma is another property of minimum counterexamples of Theorem 2.

Lemma 6. *Let G be a counterexample of Theorem 2 with minimum number of edges. Then, for each $X \subset V(G)$ with $|X| = 3$, we have that $G \setminus X$ has at most 2 components such that each component has a vertex of degree at least 3 in G.*

Our proof of Lemma 6 is provided in Section 4.

On the other hand, in order to prove Theorem 2, we use two known results: one is about graphs with no K_5 minor, and the other is about dynamic colorings. Halin [11,12] proved that every non-planar graph with no K_5 minor contains a subdivision of V_8 (see Fig. 1) as a subgraph or it has a set X of three vertices such that $G \setminus X$ has at least three components. (A slightly stronger version was proved by Kézdy and McGuinness [14, Theorem 3.6] later.) Moreover, Halin observed the following theorem since a 3-connected graph not containing K_5 minor but containing V_8 minor is isomorphic to V_8.

Theorem 7 (*Halin [11,12]*)**.** *Every 3-connected nonplanar graph with no K_5 minor is isomorphic to V_8 (see Fig. 1) or has a set X of three vertices such that $G \setminus X$ has at least 3 components.*