Discrete Applied Mathematics 206 (2016) 122–151

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Core-based criterion for extreme supermodular functions

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ARTICLE INFO

Article history:
Received 30 October 2014
Accepted 24 January 2016
Available online 3 March 2016

Keywords:
Supermodular function
Submodular function
Core
Conditional independence
Generalized permutohedron
Indecomposable polytope

ABSTRACT

We give a necessary and sufficient condition for extremality of a supermodular function based on its min-representation by means of (vertices of) the corresponding core polytope. The condition leads to solving a certain simple linear equation system determined by the combinatorial core structure. This result allows us to characterize indecomposability in the class of generalized permutohedra. We provide an in-depth comparison between our result and the description of extremality in the supermodular/submodular cone achieved by other researchers.

1. Introduction

Supermodular functions have been investigated in various branches of discrete mathematics, namely in connection with cooperative games [33], conditional independence structures [34] and generalized permutohedra [25]. Submodular functions, their mirror images, were studied in matroid theory [23] and combinatorial optimization [11,32]. Throughout this paper we regard a supermodular function as a real function defined on the power set of a finite set of variables and satisfying the supermodularity law. As the set of (suitably standardized) supermodular functions forms a pointed polyhedral cone in a finite-dimensional space, it has a finite number of extreme rays. Characterizing extremality in the supermodular cone is of vital importance for understanding its structure. This task is solved in this paper: our main result, Theorem 5, provides a necessary and sufficient condition for extremality of a supermodular function. The condition has the form of a simple criterion based on solving a system of linear equations.

The research on extreme supermodular functions has been ongoing in a number of different mathematical disciplines. Let us mention just a few of them to summarize the motivation for this paper and to recall some previous results related to the supermodular/submodular cone. Our list is by no means exhaustive.

1. *Coalition games.* The mathematical model of a cooperative game in a coalitional form is due to von Neumann and Morgenstern [37]. *Convex games* were introduced as supermodular functions on the class of all coalitions by Shapley [33]. Interestingly enough, in the 1972 paper Shapley enumerates all the extreme rays of the cone of convex games over the four-player set. Nonetheless, he claims that "For larger n, little is known about the set of all extremals". A lot of effort was exerted to describe the geometrical structure of the *core*, which is a non-empty polytope associated with any (convex) game. The core concept is also among the crucial instruments employed in this paper. Namely we rely on the

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http://dx.doi.org/10.1016/j.dam.2016.01.019
0166-218X/

characterization of the vertices of the core achieved by Shapley [33] and Weber [40]. The properties of the core allow one to characterize convex games within the set of all the coalition games; see [14,6]. Kuipers et al. [17] provided a facet description of the supermodular cone. Danilov and Koshevoy [6] employed the Möbius inversion in order to express the core of a (not necessarily supermodular) coalition game as a signed Minkowski sum of standard simplices.

2. *Conditional independence structures.* Conditional independence structures arising in discrete probabilistic framework belong to a wider class of *structural* (conditional) *independence models*, which can be interpreted as models produced by supermodular functions; see [34, § 5.4.2]. In fact, the lattice of structural independence models is anti-isomorphic to the face lattice of the cone of supermodular functions. Thus, characterizing the extreme rays of the cone of standardized supermodular functions can have both the theoretical significance in characterizing co-atoms of the lattice of structural models and some practical consequences for conditional independence implication; see [3, § 4.1]. Extreme supermodular functions also establish quite an important class of inequalities that are used in (integer) linear programming approach to learning Bayesian network structure; see [36, § 3.1] or [5, § 7]. There were attempts to classify extreme supermodular functions and the operations with them [35,15]; see also the open problems from § 9.1.2 of [34].
3. *Generalized permutohedra.* These polytopes were introduced by Postnikov [25,26] as the polytopes obtainable by moving vertices of the usual permutohedron while the directions of edges are preserved. The connection of generalized permutohedra to supermodular and submodular functions has been indicated by Doker [8]. Morton [21] earlier discussed the role of generalized permutohedra directly in the study of conditional independence structures. The class of generalized permutohedra appears to coincide with the class of core polytopes for supermodular functions. This allows us to derive as a by-product of our result a necessary and sufficient condition for a generalized permutohedron to be *indecomposable* in the sense of Meyer [20]. Although the task to characterize indecomposable generalized permutohedra has not been raised in the literature, we hope it is relevant to this topic.
4. *Combinatorial optimization and matroids.* The importance of *submodular functions*, which can be viewed as mirror images of supermodular functions, has widely been recognized in combinatorial optimization; see [11], for example. In fact, the core polytopes correspond to the so-called *base polyhedra* for submodular functions. In this context, a non-decreasing submodular function is called a *rank function* of a polymatroid. As noted by Schrijver [32, p. 781], already Edmonds [9] raised the problem of determining the extreme rays of the cone of rank functions of polymatroids. Nguyen [22] gave a criterion to recognize whether a rank function of a *matroid* [23] generates an extreme ray of that cone. One of his followers was Kashiwabara [16] who provided more general sufficient conditions for extremality of certain integer-valued submodular functions in terms of their combinatorial properties. A few other researchers studied the submodular functions in different frameworks. These functions have wide applications in computer science as explained by Živný et al. in [42, § 1.3], who also discussed a conjecture on the extreme rays of the cone of Boolean submodular functions, which was raised in supermodular context by Promislow and Young [27].
5. *Imprecise probabilities.* Theory of imprecise probabilities deals with generalized models of uncertainty reaching beyond the usual assumption in probability theory, namely the additivity axiom. One of the basic concepts in this theory is that of a *coherent lower probability*, which corresponds to a well-known game-theoretical notion of an *exact game*; see Corollary 3.3.4 in the book [39] by Walley. Similarly, the concepts of a *credal set* and of a *2-monotone lower probability* are the counterparts of the concepts of a core polytope and of a (normalized) non-negative supermodular game, respectively. Quaeghebeur and de Cooman [28] raised the question of characterizing the extreme lower probabilities and computed some of them for a small number of variables. Even a more general task has been addressed in the literature: the characterization of *extreme lower previsions* given by De Bock and de Cooman [7] relates them to indecomposable compact convex sets in a finite-dimensional space.

In this study we proceed without having any particular domain of application in mind, but being aware of the presence of this topic on the crossroad of many different disciplines mentioned above. We make an ample use of techniques and results from coalition game theory and finite-dimensional convex geometry. The key technical tool presented herein is a transformation which associates a certain polytope, called the *Weber set*, with every game. The point is that the Weber set of a supermodular game coincides with its core as defined in coalition game theory. Our main result, Theorem 5, basically asserts that a supermodular game is extreme if and only if the combinatorial structure of its core fully determines its geometry. The combinatorial concept of a "core structure" we use here has already appeared in the literature: it was formally defined by Kuipers et al. [17].

The close relation between supermodular functions and generalized permutohedra pervades this paper. This correspondence is realized via a min-representation of a supermodular function based on its core. Our Corollary 11 shows that the cores of supermodular games coincide with generalized permutohedra. As a consequence of the extremality characterization we provide a necessary and sufficient condition for indecomposability of generalized permutohedra (see Theorem 14).

The article is structured as follows. We fix our notation and terminology in Section 2. In particular, we introduce the key notions of payoff-array transformation and the Weber set there. Moreover, we formulate fundamental Lemma 1 and explain how to recover the vertices of the core for a supermodular game. Our main result, Theorem 5, is formulated in Section 3. The use of the main theorem is demonstrated by some examples and the interpretation of our criterion of extremality is discussed. The proof of Theorem 5 is postponed to Section 4. A close connection between supermodular games and generalized permutohedra is revealed in Section 5. Sections 6 and 7 contain an extensive and detailed discussion on previous results on extremality criteria for supermodular and submodular functions from the literature. In order to