Discrete Applied Mathematics 206 (2016) 165–171

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

# A minimax result for perfect matchings of a polyomino graph☆

Xiangqian Zhou, Heping Zhang*

*School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, PR China*

ARTICLE INFO

*Article history:*
Received 1 August 2015
Received in revised form 10 January 2016
Accepted 26 January 2016
Available online 26 February 2016

*Keywords:*
Polyomino graph
Perfect matching
Resonant set
Forcing number

ABSTRACT

Let $G$ be a plane bipartite graph that admits a perfect matching. A forcing set for a perfect matching $M$ of $G$ is a subset $S$ of $M$ such that $S$ is not contained by other perfect matchings of $G$. The minimum cardinality of forcing sets of $M$ is called the forcing number of $M$, denoted by $f(G, M)$. Pachter and Kim established a minimax result: for any perfect matching $M$ of $G$, $f(G, M)$ is equal to the maximum number of disjoint $M$-alternating cycles in $G$. For a polyomino graph $H$, we show that for every perfect matching $M$ of $H$ with the maximum or second maximum forcing number, $f(H, M)$ is equal to the maximum number of disjoint $M$-alternating squares in $H$. This minimax result does not hold in general for other perfect matchings of $H$ with smaller forcing number.



## 1. Introduction

A polyomino graph is a connected finite subgraph of the infinite square grid in the plane such that each finite face is bounded by a (unit) square and each edge belongs to at least one square [5,20,23]. Polyomino graphs are models of many interesting combinatorial subjects, such as hypergraph [3], domination problem [7,8], rook polynomials [14], etc. The dimer problem in crystal physics is to count perfect matchings of polyomino graphs [10].

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A *perfect matching or* 1-*factor* of $G$ is a set of disjoint edges covering all vertices of $G$. Let $M$ be a perfect matching of $G$. A cycle $C$ of $G$ is said to be $M$-*alternating* if the edges of $C$ appear alternately between $M$ and $E(G)\backslash M$. For a subgraph $F$ of $G$, let $G - F$ denote the graph $G - V(F)$.

Let $G$ be a plane graph. The boundary of the infinite face of $G$ is called the *boundary* of $G$, denoted by $\partial(G)$. The boundary of a finite face of $G$ is called a *facial cycle* if it is a cycle of $G$. A set $R$ of disjoint even facial cycles of $G$ is called a *resonant set* if $G$ has a perfect matching $M$ such that all cycles in $R$ are $M$-alternating [12], equivalently, $G - R$ either has a perfect matching or is an empty graph (no vertices). A resonant set of $G$ is said to be *maximum* if it has the maximum cardinality. The size of a maximum resonant set of $G$ is called the *Clar number* of $G$, denoted by $cl(G)$. The Clar number appeared in Clar's aromatic sextet theory of benzenoid hydrocarbons [6], and was extended to carbon fullerenes [18,22] and general plane bipartite graphs [1].

The forcing number or the innate degree of freedom for a perfect matching of a graph was originally introduced for benzenoid systems by Harary et al. [9] and Klein and Randić [11].

Let $G$ be a graph with a perfect matching $M$. A subset $S \subseteq M$ is called a *forcing set* of $M$ if it is not contained by other perfect matchings of $G$. It is known that $S$ is a forcing set of $M$ if and only if each $M$-alternating cycle of $G$ contains at least one

☆ This work is supported by NSFC (Grant Nos. 11371180 and 11401279).
* Corresponding author.
*E-mail addresses:* zhouxiangqian0502@126.com (X. Zhou), zhanghp@lzu.edu.cn (H. Zhang).

http://dx.doi.org/10.1016/j.dam.2016.01.033
0166-218X/

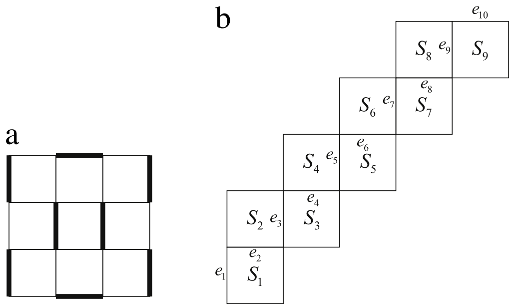


**Fig. 1.** (a) The $3 \times 3$ chessboard, and (b) a zigzag chain.

edge in $S$. The *forcing number* of $M$, denoted by $f(G, M)$, is the smallest cardinality over all forcing sets of $M$. The *maximum* (resp. *minimum*) *forcing number* of $G$ is the maximum (resp. minimum) value of forcing numbers of all perfect matchings of $G$, denoted by $F(G)$ (resp. $f(G)$). For the topic, we may refer to a survey [4] and other references [2,15,17,27].

For planar bipartite graphs, Pachter and Kim revealed a minimax fact that connects the forcing number of a perfect matching and its alternating cycles as follows.

**Theorem 1.1** ([15]). *Let $M$ be a perfect matching of a plane bipartite graph $G$. Then $f(G, M) = c(M)$, where $c(M)$ denotes the maximum number of disjoint $M$-alternating cycles in $G$.*

A hexagonal system can be formed by a cycle of an infinite plane hexagonal lattice and its interior. In 1985, Zheng and Chen [26] showed that the graph obtained from a hexagonal system by deleting any maximum resonant set has a unique perfect matching. By combining Theorem 1.1 and Zheng and Chen's result, Xu et al. [17] obtained the following equality between that the maximum forcing number and the Clar number in a hexagonal system.

**Theorem 1.2** ([17]). *Let $H$ be a hexagonal system with a perfect matching. Then $cl(H) = F(H)$.*

Zheng and Chen's result relies on the fact [26]: if the graph obtained from a hexagonal system $H$ by deleting the boundary and a resonant set $K$ whose hexagons are disjoint with the boundary has a perfect matching or is empty, then the Clar number of $H$ is at least $|K| + 1$. By increasing the lower bound by one, the present authors obtained a stronger result as follows.

**Theorem 1.3** ([27]). *Let $H$ be a hexagonal system with a perfect matching. For every perfect matching $M$ of $H$ such that $f(H, M) = F(H)$, $H$ has $F(H)$ disjoint $M$-alternating hexagons.*

We also showed that Theorem 1.2 holds for polyomino graphs [25]. For a polyomino graph $H$ with a perfect matching $M$, let $h(M)$ denote the maximum number of disjoint $M$-alternating squares in $H$. Theorem 1.1 implies $f(H, M) = c(M) \geq h(M)$. The second equality does not necessarily hold in general. As an example, consider the $3 \times 3$ chessboard illustrated in Fig. 1(a). The bold edges in the chessboard constitute a perfect matching $M'$ whose forcing number equals 2, whereas the chessboard has only one $M'$-alternating square.

In this article, however we show that $f(H, M) = h(M)$ always holds for every perfect matching $M$ of a polyomino graph $H$ whose forcing number reaches the maximum or second maximum value.

**Theorem 1.4.** *Let $H$ be a polyomino graph with a perfect matching. Then for every perfect matching $M$ of $H$ such that $f(H, M) = F(H)$ or $F(H) - 1$, the forcing number of $M$ is equal to the maximum number of disjoint $M$-alternating squares in $H$.*

To prove this main result, we intend to improve the crucial Lemma 2.2 of [25] to obtain Lemma 2.1 in Section 2 (in fact we increase the previous lower bound [25] on the Clar number $cl(H)$ of a polyomino graph $H$ by two). Based on the lemma, in Section 3 we describe clearly structural properties for a maximum set of disjoint $M$-alternating cycles of $H$ for any perfect matching $M$ with $f(H, M) = F(H)$ or $F(H) - 1$, then we give a proof of this main result.

## 2. A sharp lower bound on the Clar number

Let $H$ be a polyomino graph. A square of $H$ is called an *internal square* if it is disjoint with the boundary $\partial(H)$ and an *external square* otherwise. We may assume all polyomino graphs are embedded in the plane such that the edges are either vertical or horizontal. We state a crucial lemma as follows.

**Lemma 2.1.** *Let $H$ be a 2-connected polyomino graph and let $R$ be a nonempty resonant set of $H$ consisting of internal squares. Suppose $H - R - \partial(H)$ has a perfect matching or is an empty graph. Then $cl(H) \geq |R| + 3$.*