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Clique cover products and unimodality of independence polynomials[☆]

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ABSTRACT

Given two graphs G and H, assume that $\mathscr{C} = \{C_1, C_2, \ldots, C_q\}$ is a clique cover of G and U is a subset of $V(H)$. We introduce a new graph operation called the clique cover product, denoted by $G^{\mathscr{C}} \star H^U$, as follows: for each clique $C_i \in \mathscr{C}$, add a copy of the graph H and join every vertex of C_i to every vertex of U. We prove that the independence polynomial of $G^{\mathscr{C}} \star H^U$

$$I(G^{\mathscr{C}} \star H^U; x) = [I(H; x)]^q I\left(G; \frac{xI(H-U; x)}{I(H; x)}\right),$$

which generalizes some known results on independence polynomials of the compound graph introduced by Song, Staton and Wei, the corona and rooted products of graphs obtained by Gutman and Rosenfeld, respectively. Based on this formula, we show that the clique cover product of some graphs preserves symmetry, unimodality, log-concavity or reality of zeros of independence polynomials. As applications we derive several known facts and solve some open unimodality conjectures and problems in a simple and unified manner.

1. Introduction

1.1. Independence polynomials of graphs

For the graph theoretical terms used but not defined, we follow Bondy and Murty [4]. Let $G = (V(G), E(G))$ be a finite and simple graph. An *independent set* in a graph G is a set of pairwise non-adjacent vertices. A *maximum independent set* in G is a largest independent set and its size is denoted $\alpha(G)$. Let $i_k(G)$ denote the number of independent sets of cardinality k in G. Then its generating function

$$I(G; x) = \sum_{k=0}^{\alpha(G)} i_k(G)x^k, \quad i_0(G) = 1$$

is called the *independence polynomial* of G (Gutman and Harary [21]).

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1.2. Unimodality and log-concavity of polynomials

Let $(a_k)_{k=0}^n$ be a sequence of positive real numbers. It is called *unimodal* if there is some m, called a *mode* of the sequence, such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

It is called *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq n-1$. It is called *symmetric* if $a_k = a_{n-k}$ for $0 \leq k \leq n$.

Clearly, a log-concave sequence of positive numbers is unimodal (see, e.g., Brenti [6]). If $(a_k)_{k=0}^n$ is unimodal (log-concave, symmetric, respectively), then we also say that its generating function $\sum_{k=0}^n a_k x^k$ is *unimodal* (*log-concave*, *symmetric*, respectively). A mode of the sequence $(a_k)_{k=0}^n$ is also called a mode of $\sum_{k=0}^n a_k x^k$. Unimodality and log-concavity problems arise naturally in many branches of mathematics and have been extensively investigated. See Stanley's survey [38] and Brenti's supplement [7] for various results on unimodality and log-concavity. It is well known that if the generating function $\sum_{k=0}^n a_k x^k$ has only real zeros, then by Newton's inequalities the sequence $(a_k)_{k=0}^n$ is log-concave and unimodal (see [23, p. 104]). In addition, this is a classical approach to demonstrating the log-concavity in combinatorics since polynomials arising from combinatorics are often real-rooted, see [5,33,40] for instance.

1.3. Unimodality problems of independence polynomials

There are many interesting unimodality problems in graph theory. For example, it is well known that the matching polynomial of a graph has only real zeros [24]. The long-standing open problems on the unimodality (Read [35, p. 68]) and log-concavity (Welsh [42, p. 266]) of the chromatical polynomial of a graph has recently been solved in [26]. The log-concavity problems of genus polynomials of graphs [12,18,19] are also interesting. On the other hand, unimodality problems and zeros of independence polynomials have been investigated, e.g., see [1–3,8–11,13,14,22,25,28–32,34,41,43] for an extensive literature in recent years. In fact, the independence polynomial can be regarded as a generalization of the matching polynomial because the matching polynomial of a graph and the independence polynomial of its line graph are identical. Wilf asked whether the independence polynomials are also unimodal. However, Alavi, Malde, Schwenk, Erdős [1] gave a negative example. A natural problem is the following.

Problem 1.1. For what kind of graphs, are their independence polynomials unimodal or log-concave ?

In particular, in [1] they conjectured.

Conjecture 1.1. *The independence polynomial of any tree or forest is unimodal.*

The independence polynomials for certain special classes of graphs are unimodal and even have only real zeros (*e.g.*, claw-free graphs, see [14]). Although the independence polynomial of almost every graph of order n has a nonreal zero, the average independence polynomials always have all real and simple zeros [11]. Hence an interesting problem naturally arises.

Problem 1.2 (*[8]*)**.** When does the independence polynomial of a graph have only real zeros ?

The symmetries of the matching polynomial and the characteristic polynomial of a graph were observed (see [17,27] for instance). Thus, we naturally study the symmetric independence polynomials. A few ways to construct graphs having symmetric independence polynomials were given in [39]. However, the following general problem is still open.

Problem 1.3. When is the independence polynomial of a graph symmetric ?

1.4. Clique cover product and its independence polynomial

To study above Problems and Conjectures for independence polynomials, we often need to know many information about the independence polynomials. In general, it is an NP-complete problem to determine the independence polynomial, since evaluating $\alpha(G)$ is an NP-complete problem [16]. Thus, a classical question is how to compute the independence polynomial of a graph. An approach to computing the independence polynomial of a graph is in term of those of its subgraphs. For instance, one can deduce (e.g., Gutman and Harary [21]) that

$$I(G_1 \cup G_2; x) = I(G_1; x)I(G_2; x), \qquad I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1,$$

where $G_1 + G_2$ denotes the *join* of two disjoint graphs G_1 and G_2, with $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$ as the edge set and $V(G_1) \cup V(G_2)$ as the vertex set. It is known that there are many important operations of graphs in graph theory. Motivated by the above mentioned examples, one may further ask which operation of graphs is good to compute the independence polynomial.