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Note

Largest domination number and smallest independence number of forests with given degree sequence

Michael Gentner [a], Michael A. Henning [b], Dieter Rautenbach [a,*]

[a] *Institute of Optimization and Operations Research, Ulm University, Ulm, Germany*

[b] *Department of Mathematics, University of Johannesburg, Auckland Park, 2006, South Africa*

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ABSTRACT

For a sequence d of non-negative integers, let $\mathcal{F}(d)$ be the set of all forests whose degree sequence is d. We present closed formulas for $\gamma_{\max}^{\mathcal{F}}(d) = \max\{\gamma(F) : F \in \mathcal{F}(d)\}$ and $\alpha_{\min}^{\mathcal{F}}(d) = \min\{\alpha(F) : F \in \mathcal{F}(d)\}$ where $\gamma(F)$ and $\alpha(F)$ are the domination number and the independence number of a forest F, respectively.

1. Introduction

We consider finite, simple, and undirected graphs, and use standard terminology. For a sequence d of non-negative integers, let $\mathcal{G}(d)$ be the set of all graphs with degree sequence d. Similarly, let $\mathcal{F}(d)$ be the set of all forests with degree sequence d. For some graph parameter π and an optimization goal opt $\in \{\min, \max\}$, let

$$\pi_{\rm opt}(d) = {\rm opt}\{\pi(G) : G \in \mathcal{G}(d)\} \quad \text{and} \quad \pi_{\rm opt}^{\mathcal{F}}(d) = {\rm opt}\{\pi(F) : F \in \mathcal{F}(d)\}.$$

Note that for every graph G with degree sequence d, the values of $\pi_{\min}(d)$ and $\pi_{\max}(d)$ are the best possible lower and upper bounds on $\pi(G)$ that only depend on the degree sequence of G.

In the present paper we focus on two of the most prominent computationally hard graph parameters; the domination number $\gamma(G)$ and the independence number $\alpha(G)$ of a graph G. Many of the well known bounds [1–4,6,8–11,15,16] on these two parameters depend only on the degree sequence, or on derived quantities such as the order, the size, the minimum degree, and the maximum degree, which motivates the study of $\pi_{\min}(d)$ and $\pi_{\max}(d)$. Rao [12] obtained the surprising result that $\alpha_{\max}(d)$ can be determined efficiently for every degree sequence d (cf. also [7,13,14,17]). In [5] we showed that $\gamma_{\min}(d)$ can be determined efficiently for degree sequences with bounded entries, and we gave closed formulas for $\gamma_{\min}^{\mathcal{F}}(d)$ as well as for $\alpha_{\max}^{\mathcal{F}}(d)$.

Bauer et al. [1] conjectured that $\alpha_{\min}(d)$ is computationally hard, and we [5] believe that the same is true for $\gamma_{\max}(d)$. Therefore, for these last two parameters, we focus on the more restricted case of forests. Our main results are closed formulas for $\gamma_{\max}^{\mathcal{F}}(d)$ and $\alpha_{\min}^{\mathcal{F}}(d)$. Note that for some degree sequences of forests, there are exponentially many non-isomorphic

* Corresponding author.
E-mail addresses: michael.gentner@uni-ulm.de (M. Gentner), mahenning@uj.ac.za (M.A. Henning), dieter.rautenbach@uni-ulm.de (D. Rautenbach).

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realizations. Therefore, the simple linear time algorithms that determine the domination number and the independence number of a given forest do not lead to an efficient algorithm that determines $\gamma^{\mathcal{F}}_{\max}(d)$ and $\alpha^{\mathcal{F}}_{\min}(d)$.

Let d be a sequence $(d_1, \ldots, d_n)$ of n non-negative integers. The sequence d is non-increasing if $d_1 \geq d_2 \geq \cdots \geq d_n$. For a non-negative integer i, let $n_i(d)$ and $n_{\geq i}(d)$ be the numbers of entries of d that are equal to i and at least i, respectively. It is well-known that d is the degree sequence of some forest if and only if $\sum_{i=1}^n d_i$ is an even number at most $2(n - n_0(d)) - 2$. More specifically, if $\sum_{i=1}^n d_i = 2(n - n_0(d)) - 2c$ for some positive integer c, then every forest with degree sequence d has $n_0(d)$ isolated vertices and c further non-trivial components. In particular, if all entries of d are positive, then d is the degree sequence of a tree if and only if $\sum_{i=1}^n d_i = 2n - 2$.

Let G be a graph. For a non-negative integer i, let $V_i(G)$ and $V_{\geq i}(G)$ be the sets of vertices of G of degree i and at least i, respectively. A vertex of degree at least 2 with a neighbor of degree 1 is a support vertex. A dominating set of G is a set D of vertices of G such that every vertex of G that does not lie in D has a neighbor in D, and the domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. An independent set in G is a set of pairwise non-adjacent vertices of G, and the independence number $\alpha(G)$ of G is the maximum cardinality of an independent set in G.

2. Results

We split this section into two subsections, the first containing two preparatory lemmas and the second containing our main results.

2.1. Two preparatory lemmas

Two well known and simple facts, which will be used below, are the following: Every tree of order at least 3 has a minimum dominating set that contains all support vertices. Every tree of order at least 3 has a maximum independent set that contains all vertices of degree 1.

Lemma 1. *If T is a tree of order n, then there is a set D of at most $\left\lceil \frac{n-2}{3} \right\rceil$ vertices of T such that every vertex u of T that has degree at least 2 and does not belong to D has a neighbor in D.*

Proof. We prove the statement by induction on the order n. If $n \leq 2$, then T has no vertex of degree at least 2, and $D = \emptyset$ has the desired properties. Now, let $n \geq 3$. Let $u_0 u_1 \ldots u_\ell$ be a longest path in T. If $\ell = 2$, then T is a star of order at least 3 with a center vertex u, and $D = \{u\}$ has the desired properties. Hence, we may assume that $\ell \geq 3$. Let T' be the component of $T - u_2 u_3$ that contains u_3. Clearly, the order n' of T' satisfies $n' \leq n - 3$. By induction, there is a set D' of at most $\left\lceil \frac{n'-2}{3} \right\rceil$ vertices of T' such that every vertex u of T' that has degree at least 2 (in T') and does not belong to D' has a neighbor in D'. Now, the set $D = D' \cup \{u_2\}$ contains at most $\left\lceil \frac{n'-2}{3} \right\rceil + 1 \leq \left\lceil \frac{n-2}{3} \right\rceil$ vertices and has the desired properties. Note that u_3 might have degree less than 2 in T' but is adjacent to $u_2 \in D$ in T. $\square$

Lemma 2. *If $d = (d_1, \ldots, d_n)$ is a non-increasing sequence of positive integers such that $\sum_{i=1}^n d_i = 2n - 2c$ for some positive integer c and $n_1(d) \leq n_{\geq 2}(d)$, then there is a forest F with c components and degree sequence d such that*

(i) *there are exactly $n_1(d)$ support vertices in F each of which is adjacent to exactly one vertex of degree 1,*
(ii) *the vertices in $V_{\geq 2}(F)$ that are not support vertices are all of degree 2, and induce a path P of order $n - 2n_1(d)$,*
(iii) *$\gamma(F) = \left\lceil \frac{n+n_1(d)-2}{3} \right\rceil$, and $\alpha(F) = \left\lceil \frac{n}{2} \right\rceil$.*

Proof. Since $\sum_{i=1}^n d_i = 2n - 2c$, we obtain

$$\begin{aligned} 2c &= 2n - \sum_{i=1}^{n} d_i \\ &= -\sum_{i=1}^{n} (d_i - 2) \\ &= -\sum_{i=1}^{n_{\geq 3}(d)} (d_i - 2) - \sum_{i=n_{\geq 3}(d)+1}^{n_{\geq 2}(d)} (d_i - 2) - \sum_{i=n_{\geq 2}(d)+1}^{n} (d_i - 2) \\ &= -\sum_{i=1}^{n_{\geq 3}(d)} (d_i - 2) + n_1(d), \end{aligned}$$

which implies

$$n_1(d) = 2c + \sum_{i=1}^{n_{\geq 3}(d)} (d_i - 2) \geq 2c + n_{\geq 3}(d). \tag{1}$$