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Note

On the uniqueness of some girth eight algebraically defined graphs

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A R T I C L E   I N F O

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Dedicated to the memory of Vasyl Dmytrenko (1961–2013)

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A B S T R A C T

Let $\mathbb{F}$ be a field. For a polynomial $f \in \mathbb{F}[x, y]$, we define a bipartite graph $\Gamma_{\mathbb{F}}(f)$ with vertex partition $P \cup L$, $P = \mathbb{F}^3 = L$, and $(p_1, p_2, p_3) \in P$ is adjacent to $[l_1, l_2, l_3] \in L$ if and only if

$$p_2 + l_2 = p_1 l_1 \quad \text{and} \quad p_3 + l_3 = f(p_1, l_1).$$

It is known that the graph $\Gamma_{\mathbb{F}}(xy^2)$ has no cycles of length less than eight. The main result of this paper is that $\Gamma_{\mathbb{F}}(xy^2)$ is the only graph $\Gamma_{\mathbb{F}}(f)$ with this property when $\mathbb{F}$ is an algebraically closed field of characteristic zero; i.e. over such a field $\mathbb{F}$, every graph $\Gamma_{\mathbb{F}}(f)$ with no cycles of length less than eight is isomorphic to $\Gamma_{\mathbb{F}}(xy^2)$. We also prove related uniqueness results for some polynomials f over infinite families of finite fields.

1. Introduction

It is well known that the greatest number of edges in a graph of order n of girth (i.e. the shortest cycle length) eight is of magnitude $n^{1+\frac{1}{3}}$. The upper bound comes from Bondy–Simonovits [2] and the lower bound from generalized quadrangles (which will be defined later) or their subgraphs. The primary motivation for this paper is the desire to prove the uniqueness (in a certain sense) of the existing constructions for the lower bound.

For definitions related to graphs, we refer the reader to Bollobás [1]. Our primary object of study in this paper is defined as follows. For a field $\mathbb{F}$ and two polynomials $f_2, f_3 \in \mathbb{F}[x, y]$, let P and L be two copies of the 3-dimensional vector space $\mathbb{F}^3$. Consider a bipartite graph $\Gamma_{\mathbb{F}}(f_2, f_3)$ with vertex partitions P and L and with edges defined as follows: for every $(p) = (p_1, p_2, p_3) \in P$ and every $[l] = [l_1, l_2, l_3] \in L$, $\{(p), [l]\} = (p)[l]$ is an edge in $\Gamma_{\mathbb{F}}(f_2, f_3)$ if

$$\begin{aligned} p_2 + l_2 &= f_2(p_1, l_1) \\ p_3 + l_3 &= f_3(p_1, l_1). \end{aligned}$$

It turns out that the graph $\Gamma_{\mathbb{F}}(xy, xy^2)$ has girth eight; furthermore, when $\mathbb{F}$ is finite, it is isomorphic to an induced subgraph of a classical generalized quadrangle of order q (see Section 6 for details).

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This brings up a natural question: is $\Gamma_{\mathbb{F}}(xy, xy^2)$ the unique (up to isomorphism) girth eight graph of the form $\Gamma_{\mathbb{F}}(f_2, f_3)$?

For some uniqueness results over finite fields of odd characteristic, see Dmytrenko [4], Dmytrenko, Lazebnik and Williford [5], Kronenthal [8], Hou, Lappano, and Lazebnik [7], and references therein. The approach in these papers was to use properties of polynomials over finite fields. In this paper we use another approach. We let $\mathbb{F}$ be an algebraically closed field of characteristic zero, for example the field of complex numbers $\mathbb{C}$. For such $\mathbb{F}$, we prove the uniqueness of $\Gamma_{\mathbb{F}}(xy, xy^2)$ for all graphs in 'close vicinity', i.e. all graphs of the form $\Gamma_{\mathbb{F}}(xy, f)$. We then prove related uniqueness results for some polynomials f over infinite families of finite fields.

The main results of this paper are as follows.

Theorem 1.1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. Suppose $f \in \mathbb{F}[x, y]$ and the graph $\Gamma_{\mathbb{F}}(xy, f)$ has girth at least eight. Then $\Gamma_{\mathbb{F}}(xy, f)$ is isomorphic to $\Gamma_{\mathbb{F}}(xy, xy^2)$.*

The following theorem is an analog of Theorem 1.1 for finite fields $\mathbb{F}_q$ of odd characteristic p and polynomials of 'small' degree. Let $M = M(p)$ be the least common multiple of the integers $1, 2, \ldots, p-2$. The function M is defined in this way so that all polynomials over $\mathbb{F}_q$ of degree at most $p-2$ have a root in $\mathbb{F}_{q^M}$.

Theorem 1.2. *Let q be a power of a prime p, $p \geq 5$. Suppose $f \in \mathbb{F}_q[x, y]$ has degree at most $p-2$ with respect to each of x and y. Then for all positive integers r, every graph $\Gamma_{q^{Mr}}(xy, f)$ of girth at least eight is isomorphic to $\Gamma_{q^{Mr}}(xy, xy^2)$.*

The prime $p = 3$ (so $M = 1$) is excluded, as it is easy to argue in this case that every $\Gamma_{q^r}(xy, f)$ has girth six.

For a polynomial $f = \sum_{0 \leq i,j \leq n} a_{ij}x^iy^j \in \mathbb{Z}[x, y]$, let $\hat{f} = \sum_{0 \leq i,j \leq n} \hat{a}_{ij}x^iy^j \in \mathbb{F}_p[x, y]$, where $\hat{a}_{ij}$ is the image of a_{ij} with respect to the canonical homomorphism $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

Corollary 1.3 (*to* Theorem 1.1). *Suppose $f \in \mathbb{Z}[x, y]$. Then there exists a positive constant $c = c(f)$ such that for every prime $p > c(f)$, there exists an integer $s = s(f, p)$ such that for all positive integers r, and $q = p^{sr}$, every graph $\Gamma_q(xy, \hat{f})$ of girth at least eight is isomorphic to $\Gamma_q(xy, xy^2)$.*

We wish to comment that in Theorem 1.1, we consider all polynomials f over a given algebraically closed field of characteristic zero. However, in Corollary 1.3, we fix one polynomial with integer coefficients and state the existence of infinitely many characteristics p, and infinitely many finite fields of characteristic p, over which an analog of Theorem 1.1 holds.

This paper is organized as follows. In Section 2 we provide a description of 4- and 6-cycles in graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$ and some isomorphisms between these graphs. In Sections 3 and 4, we present proofs of Theorems 1.1 and 1.2, respectively. In Section 5 we discuss the Lefschetz Principle and prove Corollary 1.3. In Section 6, we explain the relationship between the graphs $\Gamma_q(f_2, f_3)$ and generalized quadrangles, make some concluding remarks, and mention open problems.

2. Cycles and isomorphisms of graphs $\Gamma_{\mathbb{F}}(f_2, f_3)$

Let $\Gamma_{\mathbb{F}}(f_2, f_3)$ be the graph defined in Section 1. If two vertices a, b in a graph are adjacent, we will write $a \sim b$. Let us describe cycles of length four and six in $\Gamma_{\mathbb{F}}(f_2, f_3)$. If the graph contains a 4-cycle

$$(a_1, a_2, a_3) \sim [x_1, x_2, x_3] \sim (b_1, b_2, b_3) \sim [y_1, y_2, y_3] \sim (a_1, a_2, a_3), \tag{1}$$

then $(a_1, a_2, a_3) \sim [x_1, x_2, x_3]$ implies that $x_i = f_i(a_1, x_1) - a_i$ for $i = 2, 3$. Furthermore, $[x_1, x_2, x_3] \sim (b_1, b_2, b_3)$ implies that $b_i = f_i(b_1, x_1) - x_i = f_i(b_1, x_1) - f_i(a_1, x_1) + a_i$ for $i = 2, 3$. Similarly, we have:

$$\begin{aligned} y_i &= f_i(b_1, y_1) - f_i(b_1, x_1) + f_i(a_1, x_1) - a_i \\ a_i &= f_i(a_1, y_1) - f_i(b_1, y_1) + f_i(b_1, x_1) - f_i(a_1, x_1) + a_i. \end{aligned}$$

This implies that in order for this 4-cycle to exist, we must have

$$f_i(a_1, y_1) - f_i(b_1, y_1) + f_i(b_1, x_1) - f_i(a_1, x_1) = 0$$

for $i = 2, 3$. Similarly in order for a 6-cycle

$$\begin{aligned} (a_1, a_2, a_3) \sim [x_1, x_2, x_3] &\sim (b_1, b_2, b_3) \sim [y_1, y_2, y_3] \\ &\sim (c_1, c_2, c_3) \sim [z_1, z_2, z_3] \sim (a_1, a_2, a_3), \end{aligned} \tag{2}$$

to exist in $\Gamma_{\mathbb{F}}(f_2, f_3)$, we must have

$$f_i(a_1, z_1) - f_i(c_1, z_1) + f_i(c_1, y_1) - f_i(b_1, y_1) + f_i(b_1, x_1) - f_i(a_1, x_1) = 0.$$

To have a convenient notation for the alternating sums above, we define the following functions on the polynomial rings:

$$\begin{aligned} &\Delta_2 : \mathbb{F}[s_1, s_2] \to \mathbb{F}[t_1, t_2, t_3, t_4] \\ &\quad f(s_1, s_2) \mapsto f(t_1, t_3) - f(t_2, t_3) + f(t_2, t_4) - f(t_1, t_4), \end{aligned}$$