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Note

On the distance spread of cacti and bicyclic graphs

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ABSTRACT

We determine the unique graphs with minimum and second minimum distance spread in the class of cacti and in the class of bicyclic graphs, respectively. We also determine the unique graphs with minimum distance spread in the class of cacti with fixed number of cycles and fixed matching number, respectively.

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1. Introduction

We consider simple graphs. Let G be a connected graph of order n with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, the distance between u and v, denoted by $d_G(u, v)$, is the length of a shortest path from u to v in G. The distance matrix of G is the $n \times n$ matrix $\mathbf{D}(G) = (d_G(u, v))_{u,v\in V(G)}$. The polynomial $\det(\mathbf{D}(G) - x\mathbf{I}_n)$ is called the distance characteristic polynomial of G, where $\mathbf{I}_n$ is the identity matrix of order n. Let $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \dots \geqslant \lambda_n(G)$ be the distance spectrum of G, which is the spectrum of $\mathbf{D}(G)$. Note that $\lambda_1(G)$ is known as the distance spectral radius of G and $\lambda_n(G)$ is the least distance eigenvalue of G. The distance spread of G is defined as $s(G) = \lambda_1(G) - \lambda_n(G)$. The spread of a real symmetric matrix is the difference between its greatest and least eigenvalues, which has applications in combinatorial optimization problems [5]. The distance spread of G is just the spread of the distance matrix of G, and it is also known as the spectral diameter of the distance matrix of G, which is used as a molecular descriptor in chemoinformatics, see, e.g. [4,6].

A cactus is a connected graph in which any two cycles have at most one common vertex. Let $\mathcal{C}(n)$ be the class of cacti of order n, $\mathcal{C}(n, k)$ the class of cacti of order n with k cycles, and $\mathcal{C}^a(n)$ the class of cacti of order n with matching number a.

A connected graph G of order n with m edges is known as a tree, a unicyclic graph, and a bicyclic graph if $m = n - 1$, $m = n$, and $m = n + 1$, respectively. Let $\mathcal{B}(n)$ be the class of bicyclic graphs of order n.

There are many results about distance eigenvalues (especially for $\lambda_1(G)$ and $\lambda_n(G)$). Bose et al. [2] determined the unique graphs with minimum distance spectral radius in $\mathcal{C}(n)$ and $\mathcal{C}(n, k)$, respectively. They also determined the unique graphs with maximum distance spectral radius in $\mathcal{C}(n, k)$. Xing and Zhou [12] determined the unique graphs with minimum and second-minimum distance spectral radii in $\mathcal{B}(n)$. Paul [10] determined the unique graph with maximum distance spectral radius among bicyclic graphs containing three internally vertex disjoint paths with common end vertices. Bose et al. [3] determined the unique graph with maximum distance spectral radius among bicyclic graphs with two edge disjoint cycles. More results on the distance spectral radius may be found in the recent survey [1]. Compared with the much studied distance spectral radius, the study of the least distance eigenvalue have just begun in recent years, see [7,8,13].

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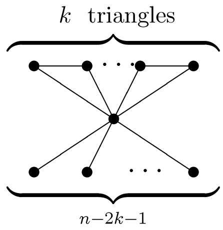

Fig. 1. Graph $C_0(n, k)$.

Yu et al. [14] determined the unique graphs with minimum distance spread among connected graphs, connected bipartite graphs, connected graphs with fixed independent number, and trees, respectively. Lin [7] determined the unique unicyclic graph and the unique connected graph with fixed clique number having minimum distance spread.

In this note, we determine the unique graphs with minimum and second-minimum distance spread in $\mathcal{C}(n)$ and in $\mathcal{B}(n)$, respectively, and the unique graphs with minimum distance spread in $\mathcal{C}(n, k)$ and in $\mathcal{C}^a(n)$, respectively. Besides, we determine the unique graphs with minimum distance spectral radius in $\mathcal{C}^a(n)$.

2. Preliminaries

A cactus is said to be a bundle if all its cycles have exactly one common vertex. Let S_n be the star of order n. Let $C_0(n, k)$ be the bundle obtained from S_n by adding k independent edges, see Fig. 1. In particular, $C_0(n, 0) = S_n$.

Lemma 2.1 ([14]). *Let G be a tree of order n. Then $s(G) \geqslant n + \sqrt{n^2 - 3n + 3}$, with equality if and only if $G \cong C_0(n, 0)$.*

Lemma 2.2 ([7]). *Let G be a connected graph of order n with diameter at least 3. Then $\lambda_n(G) \leqslant -2 - \sqrt{2}$.*

Let $\mathbf{J}_{n\times m}$ and $\mathbf{0}_{n\times m}$ be respectively all-one and all-zero $n \times m$ matrices. Let $\mathbf{J}_n = \mathbf{J}_{n\times n}$, $\mathbf{0}_n = \mathbf{0}_{n\times n}$, and $\mathbf{1}_n = \mathbf{J}_{n\times 1}$. Let $\mathbf{M}$, $\mathbf{N}$, $\mathbf{P}$, and $\mathbf{Q}$ be respectively $p \times p$, $p \times q$, $q \times p$, and $q \times q$ matrices, where $\mathbf{Q}$ is invertible. We have

$$\det\begin{pmatrix} \mathbf{M} & \mathbf{N} \\ \mathbf{P} & \mathbf{Q} \end{pmatrix} = \det \mathbf{Q} \cdot \det(\mathbf{M} - \mathbf{N}\mathbf{Q}^{-1}\mathbf{P}).$$

For $m \times n$ matrix $\mathbf{A} = (a_{ij})$ and $p \times q$ matrix $\mathbf{B} = (b_{ij})$, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ of them is the $mp \times nq$ matrix obtained from $\mathbf{A}$ by replacing each entry a_{ij} of $\mathbf{A}$ by $a_{ij}B$.

Lemma 2.3. *For $n \geqslant 5$ and $2 \leqslant k \leqslant \frac{n-1}{2}$, $\lambda_n(C_0(n, k)) = -3$.*

Proof. Let v be the vertex of maximum degree, A the set of nonpendent neighbors of v, and B the set of pendent neighbors of v in $C_0(n, k)$. Obviously, $|A| = 2k$ and $|B| = n - 2k - 1$. Note that $V(C_0(n, k)) = \{v\} \cup A \cup B$.

Case 1. $k < \frac{n-1}{2}$. Then

$$\det(\mathbf{D}(C_0(n,k)) - x\mathbf{I}_n) = \begin{vmatrix} -x & \mathbf{1}_{2k}^T & \mathbf{1}_{n-2k-1}^T \\ \mathbf{1}_{2k} & 2\mathbf{J}_{2k} - \mathbf{I}_k \otimes (\mathbf{J}_2 + (x+1)\mathbf{I}_2) & 2\mathbf{J}_{(2k)\times(n-2k-1)} \\ \mathbf{1}_{n-2k-1} & 2\mathbf{J}_{(n-2k-1)\times(2k)} & 2\mathbf{J}_{n-2k-1} - (x+2)\mathbf{I}_{n-2k-1} \end{vmatrix}$$
$$= \begin{vmatrix} -x & \mathbf{1}_k^T \otimes \mathbf{A} & \mathbf{1}_{n-2k-1}^T \\ \mathbf{1}_k \otimes \mathbf{B} & \mathbf{I}_k \otimes \mathbf{C} & \mathbf{0}_{(2k)\times(n-2k-1)} \\ (2x+1)\mathbf{1}_{n-2k-1} & \mathbf{0}_{(n-2k-1)\times(2k)} & -(x+2)\mathbf{I}_{n-2k-1} \end{vmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2x+1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} -x-3 & -1 \\ 0 & -x-1 \end{pmatrix}.$$

Let

$$\mathbf{M} = \begin{pmatrix} -x & \mathbf{1}_k^T \otimes \mathbf{A} \\ \mathbf{1}_k \otimes \mathbf{B} & \mathbf{I}_k \otimes \mathbf{C} \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} \mathbf{1}_{n-2k-1}^T \\ \mathbf{0}_{(2k)\times(n-2k-1)} \end{pmatrix},$$
$$\mathbf{P} = \begin{pmatrix} (2x+1)\mathbf{1}_{n-2k-1} & \mathbf{0}_{(n-2k-1)\times(2k)} \end{pmatrix}, \quad \text{and} \quad \mathbf{Q} = -(x+2)\mathbf{I}_{n-2k-1}.$$

Then

$$\mathbf{N}\mathbf{Q}^{-1}\mathbf{P} = \begin{pmatrix} (n-2k-1)\dfrac{2x+1}{-x-2} & \mathbf{0}_{1\times 2k} \\ \mathbf{0}_{2k\times 1} & \mathbf{0}_{2k} \end{pmatrix},$$

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