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Note

The maximum matching energy of bicyclic graphs with even girth[☆]

Gang Ma [a], Shengjin Ji [a,b,*], Qiuju Bian [a], Xia Li [a]

[a] School of Science, Shandong University of Technology, Zibo, Shandong 255049, China

[b] School of Mathematics, Shandong University, Jinan, Shandong 250100, China

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ABSTRACT

The matching energy of a graph was introduced by Gutman and Wagner in 2012 and defined as the sum of the absolute values of zeros of its matching polynomial. In this paper, we characterize the graphs with maximum matching energy among all bicyclic graphs with even girth.

1. Introduction

All graphs in this paper are finite, connected, simple and undirected. A t-vertex is a vertex of degree t. For more notations and terminologies that will be used, see [1]. Let $G = (V, E)$ be such a graph with order $|V| = n$ and size $|E| = m$. A *matching* in a graph G is a set of pairwise nonadjacent edges. A matching is called a *k-matching* if it is of size k. Let $m_k(G)$ denote the number of k-matchings of G, where $m_1(G) = m$ and $m_k(G) = 0$ for $k > \lfloor \frac{n}{2} \rfloor$ or $k < 0$. In addition, define $m_0(G) = 1$. The matching polynomial of graph G is defined as

$$\alpha(G) = \alpha(G, x) = \sum_{k\geq 0} (-1)^k m_k(G) x^{n-2k}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G. The energy of graph G [7] is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

An important tool of graph energy is the Coulson integral formula [7] (with regard to G being a tree T):

$$E(T) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln \left[\sum_{k\geq 0} m_k(T) x^{2k} \right] dx. \tag{1}$$

The graph energy has been widely studied by theoretical chemists and mathematicians. For details, see the book on graph energy [18] and reviews [8,11]. There are also some recent results about graph energy, see [14,17,19,23,24].

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* Corresponding author at: School of Science, Shandong University of Technology, Zibo, Shandong 255049, China.
E-mail addresses: math_magang@163.com (G. Ma), jishengjin2013@163.com (S. Ji), bianqiuju@163.com (Q. Bian), summer08lixia@163.com (X. Li).

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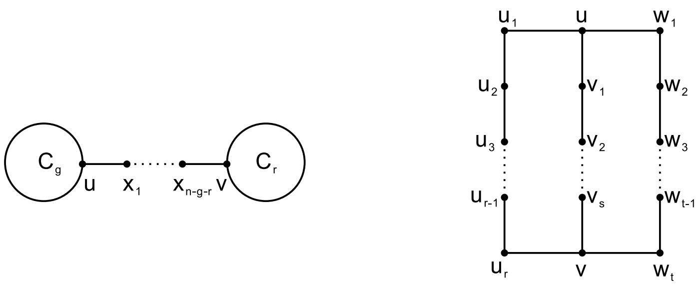

Fig. 1. Left is the graph $\infty_n(g, r)$; Right is $\theta(r, s, t)$.

In 2012, Gutman and Wagner [12] defined the matching energy of a graph G. Let G be a simple graph, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the zeros of its matching polynomial. Then

$$ME(G) = \sum_{i=1}^{n} |\mu_i|.$$

Being similar to Eq. (1), the matching energy also has a beautiful formula as follows [12]:

$$ME(G) = \frac{2}{\pi} \int_0^{+\infty} \frac{1}{x^2} \ln\left[\sum_{k\geq 0} m_k(G)x^{2k}\right] dx. \tag{2}$$

By Eq. (2) and the monotonicity of the logarithm function, the matching energy of a graph G is a monotonically increasing function of any $m_k(G)$. This means that if two graphs G and G' satisfy $m_k(G) \leq m_k(G')$ for all $k \geq 1$, then $ME(G) \leq ME(G')$. If, in addition, $m_k(G) < m_k(G')$ for at least one k, then $ME(G) < ME(G')$. It motivates the introduction of a *quasi-order* $\succeq$ as follows: If two graphs G_1 and G_2 have the same order and size, then

$$G_1 \succeq G_2 \iff m_k(G_1) \geq m_k(G_2) \quad \text{for } 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

If $G_1 \succeq G_2$ and there exists some k such that $m_k(G_1) > m_k(G_2)$, then we write $G_1 \succ G_2$. If $G_1 \succeq G_2$ we say that G_1 is *m-greater than* G_2, or G_2 is *m-smaller than* G_1. If $G_1 \succeq G_2$ and $G_2 \succeq G_1$, the graphs G_1 and G_2 are said to be *m-equivalent*, denote it by $G_1 \sim G_2$. If $G_1 \succ G_2$ we say that G_1 is *strictly m-greater than* G_2. It is easy to see that $G_1 \succeq G_2 \Longrightarrow ME(G_1) \geq ME(G_2)$ and $G_1 \succ G_2 \Longrightarrow ME(G_1) > ME(G_2)$.

A connected simple graph with n vertices and $n, n+1, n+2$ edges is called *unicyclic*, *bicyclic*, *tricyclic* graph, respectively. In [12], the authors gave some elementary results on the matching energy and obtained that $ME(S_n^+) \leq ME(G) \leq ME(C_n)$ for any unicyclic graph G, where S_n^+ is the graph obtained by adding a new edge to the star S_n. In [15], Ji et al. proved that if G is a bicyclic graph with $n \geq 10$ or $n = 8$, $ME(S_n^*) \leq ME(G) \leq ME(P_n^{4,n-4})$. In [16], the authors characterized the connected graphs (and bipartite graphs) of order n having minimum matching energy with m $(n+2 \leq m \leq 2n-4)$ edges. Especially, among all tricyclic graphs of order $n \geq 5$, $ME(G) \geq ME(S_n^{**})$, with equality if and only if $G \cong S_n^{**}$ or $G \cong K_4^{n-4}$. In [6], tricyclic graph with maximum matching energy is characterized. In [28], the authors characterized the bicyclic graph with given girth having minimum matching energy. But which graph has the maximum matching energy is not determined. Motivated by this fact, we will consider the problem in this paper. For more results about matching energy, see [2–5,13,21,22,25–27].

Denote by $\mathscr{B}_{n,g}$ the set of all connected bicyclic graphs with order n and girth g. Now define two special classes of bicyclic graphs. Let $\infty_n(g, r)$ denote the graph obtained by the coalescence of two end vertices of a path $P_{n-g-r+2}$ with one vertex of two cycles C_g and C_r respectively, and let $\theta(r, s, t)$ denote the graph obtained by fusing two triples of pendent vertices of three paths $P_{r+2}, P_{s+2}, P_{t+2}$ to two vertices, see Fig. 1. The distance of two cycles C_g and C_r in G is defined as $d_G(C_g, C_r) = \min\{d_G(x, y) | x \in V(C_g), y \in V(C_r)\}$, sometimes written as d_G for short. Note that $d_G(C_g, C_r) = 0$ if C_g and C_r have a common vertex, e.g., for $G = \infty_n(g, r)$ such that $r = n - g + 1$. Clearly, any bicyclic graph must contain either the left graph or right graph in Fig. 1 as an induced graph, which we call the *brace*. Then the set $\mathscr{B}_{n,g}$ can be partitioned into two subsets $\mathscr{B}_{n,g}^1$ and $\mathscr{B}_{n,g}^2$, where $\mathscr{B}_{n,g}^1$ is the set of all bicyclic graphs which contain a brace of the form $\infty_n(g, r)$, and $\mathscr{B}_{n,g}^2$ is the set of all bicyclic graphs which contain a brace of the form $\theta(r, s, t)$.

The main result of this paper is the following theorem which gives the graph in $\mathscr{B}_{n,g}$ with maximum matching energy, where g is even.

Theorem 1. *Suppose g is even, $g \geq 4$ and $n \geq 2g - 2$. For any $G \in \mathscr{B}_{n,g}$, then:*

(i) *if $n = 2g$ or $n \geq 2g + 2$, $ME(G) \leq ME(\infty_n(g, n - g))$ with equality if and only if $G \cong \infty_n(g, n - g)$;*

(ii) *if $n = 2g + 1$, $ME(G) \leq ME(\infty_n(g, n - g))$ with equality if and only if $G \cong \infty_n(g, n - g)$, or $G \cong \theta(0, g - 2, n - g)$;*

(iii) *if $n = 2g - 1$ or $n = 2g - 2$, $ME(G) \leq ME(\theta(0, g - 2, n - g))$ with equality if and only if $G \cong \theta(0, g - 2, n - g)$.*