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Note

Neighbor sum distinguishing total choosability of planar graphs without 4-cycles

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ABSTRACT

Let $G = (V, E)$ be a graph and ϕ be a total k-coloring of G by using the color set $\{1, \dots, k\}$. Let $\sum_\phi(u)$ denote the sum of the color of the vertex u and the colors of all incident edges of u. A k-neighbor sum distinguishing total coloring of G is a total k-coloring of G such that for each edge $uv \in E(G)$, $\sum_\phi(u) \neq \sum_\phi(v)$. By $\chi''_\Sigma(G)$, we denote the smallest value k in such a coloring of G. Pilśniak and Woźniak first introduced this coloring and conjectured that $\chi''_\Sigma(G) \leq \Delta(G) + 3$ for any simple graph G. Let $L_z (z \in V \cup E)$ be a set of lists of integer numbers, each of size k. The smallest k for which for any specified collection of such lists, there exists a neighbor sum distinguishing total coloring using colors from L_z for each $z \in V \cup E$ is called the neighbor sum distinguishing total choosability of G, and denoted by $ch''_\Sigma(G)$. In this paper, we prove that $ch''_\Sigma(G) \leq \Delta(G) + 3$ for planar graphs without 4-cycles with $\Delta(G) \geq 7$. This implies that Pilśniak and Woźniak' conjecture is true for planar graphs without 4-cycles.

1. Introduction

Only simple graphs are considered in this paper. The terminology and notation used but undefined in this paper can be found in [2]. A *plane* graph is a particular drawing of a planar graph in Euclidean plane. For a plane graph G, we denote its vertex set, edge set, face set, maximum degree and minimum degree by $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$, respectively. Let $d_G(v)$ or simply $d(v)$ denote the degree of a vertex v in G. A vertex v is called an *l-vertex* if $d(v) = l$, similarly, an *l^+-vertex* or an *l^--vertex* if $d(v) \geq l$ or $d(v) \leq l$. A face of a plane graph is said to be *incident* with all edges and vertices on its boundary. Two faces are *adjacent* if they have an edge in common. The degree of a face f of plane graph G, denoted by $d_G(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-face is a face of degree k. A k-cycle is a cycle of length k. A *triangle* is synonymous with a 3-face or a 3-cycle. A triangle $v_1v_2v_3$ is called an *(α, β, γ)-cycle*, if v_1 is an α-vertex, v_2 is a β-vertex and v_3 is a γ-vertex.

Given a graph $G = (V, E)$ and a positive integer k, a total k-coloring of G is a proper coloring $\phi : V \cup E \to \{1, \dots, k\}$, where a proper coloring means every pair of adjacent or incident elements receives different numbers. Given a total k-coloring ϕ of G, let $C_\phi(v)$ denote the set of colors of the edges incident to v and the color of v. A total k-coloring is called *adjacent vertex distinguishing* if for each edge uv, $C_\phi(u)$ is different from $C_\phi(v)$. A smallest such k is called the *adjacent vertex distinguishing total chromatic number* of G, denoted by $\chi''_a(G)$. Zhang et al. [18] put forward the following conjecture.

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Conjecture 1.1 ([18]). *For any graph G with at least two vertices, $\chi''_a(G) \leq \Delta(G) + 3$.*

Conjecture 1.1 has been proved for a few special cases, such as subcubic graphs, K_4-minor free graphs and some special planar graphs, see [3,7,13,16,17]. Recently, colorings and labelings related to sums of the colors have been studied widely, see the survey paper [12]. In a total k-coloring of G, let $\sum_\phi(v)$ denote the sum of colors of the edges incident to v and the color of v. If for each edge $uv \in E(G)$, we have $\sum_\phi(u) \neq \sum_\phi(v)$, we call such total k-coloring a *k-neighbor sum distinguishing total coloring*. The smallest number k is called *the neighbor sum distinguishing total chromatic number* of G, denoted by $\chi''_\Sigma(G)$. For neighbor sum distinguishing total colorings, we have the following conjecture due to Pilśniak and Woźniak [9].

Conjecture 1.2 ([9]). *For any graph G with at least two vertices, $\chi''_\Sigma(G) \leq \Delta(G) + 3$.*

Clearly, Conjecture 1.2 implies Conjecture 1.1 since it is easy to check that $\chi''_a(G) \leq \chi''_\Sigma(G)$. Pilśniak and Woźniak [9] proved that Conjecture 1.2 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. Dong and Wang [6] showed that Conjecture 1.2 holds for sparse graphs. Li et al. [8] confirmed this conjecture for K_4-minor free graphs. By using the famous Combinatorial Nullstellensatz, Ding et al. [5] proved that $\chi''_\Sigma(G) \leq 2\Delta(G) + \mathrm{col}(G) - 1$, where $\mathrm{col}(G)$ is the coloring number of G. Later Ding et al. [4] improved this bound to $\Delta(G) + 2\mathrm{col}(G) - 2$. In [10], Qu et al. proved that Conjecture 1.2 holds for planar graphs with maximum degree at least 11. Recently, Wang et al. confirmed the conjecture for some special planar graphs with maximum degree at least 7, see [14,15].

For a given graph G, let $L_z(z \in V \cup E)$ be any set of list of integer numbers, each of size k. If for any specified collection of such lists, there exists a neighbor sum distinguishing total coloring of G using colors from L_z for each $z \in V \cup E$, we call such coloring a *k-neighbor sum distinguishing list total coloring*, the smallest k is called *the neighbor sum distinguishing total choosability* of G, and denoted by $ch''_\Sigma(G)$. In [11], Qu et al. proved that $ch''_\Sigma(G) \leq \Delta(G) + 3$ for planar graphs with maximum degree at least 13. In this paper, we studied the neighbor sum distinguishing total choosability of planar graphs and proved the following result.

Theorem 1.1. *Let G be a planar graph without 4-cycles and $\Delta(G) \geq 7$. Then $ch''_\Sigma(G) \leq \Delta(G) + 3$.*

Clearly, $\chi''_\Sigma(G) \leq ch''_\Sigma(G)$, so the result above holds also for $\chi''_\Sigma(G)$. Our approach is based on the Combinatorial Nullstellensatz, discharging method and some other tricks, which have been widely used in coloring theory.

2. Preliminaries

Let G be a plane graph without 4-cycles, then the following configurations are excluded from G. This obvious fact will be frequently used.

(C1) A four face;

(C2) A triangle adjacent to a triangle.

In order to prove the main result, we need some lemmas.

Lemma 2.1 ([11]). *Suppose m, n are positive integers with $m \leq n$, L_i is a set of at least n integers for each $i \in \{1, \ldots, m\}$, and let $T_m(L_1, \ldots, L_m) = \{\sum_{i=1}^m x_i | x_i \in L_i, i \neq j \Longrightarrow x_i \neq x_j\}$. Then $|T_m(L_1, \ldots, L_m)| \geq mn - m^2 + 1$.*

Lemma 2.2 ([1]). *Let $\mathscr{F}$ be an arbitrary field, and let $P = P(x_1, \ldots, x_n)$ be a polynomial in $\mathscr{F}[x_1, \ldots, x_n]$. Suppose the degree $\deg(P)$ of P equals $\sum_{i=1}^n k_i$, where each k_i is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{k_i}$ in P is non-zero. Then if $S_1, \ldots, S_n$ are subsets of $\mathscr{F}$ with $|S_i| > k_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $P(s_1, \ldots, s_n) \neq 0$.*

3. Proof of main result

Let L_z (for all $z \in V \cup E$) be any given set of lists of integer numbers, each of size k, where $k = \Delta(G) + 3$. For simplicity, we use "k-nsd list total coloring" to denote "k-neighbor sum distinguishing list total coloring". Let ϕ be a k-nsd list total coloring of planar graph G without 4-cycles with $\Delta(G) \geq 7$. Assume that $u \in V(G)$ with $d(u) \leq 3$, it is easy to see that u has at most 3 adjacent vertices and 3 incident edges, and the sum obtained at u must be distinct from 3 sums at the adjacent vertices of u. So u has at most 9 forbidden colors. Since $|L_u| = k \geq 10$, we may first erase the color of u and recolor it finally. In other words, we will omit the recoloring for all 3^--vertices in the following discussion.

Our proof proceeds by *reduction and absurdum*. Assume that G is a counterexample to Theorem 1.1 such that $|V(G)| + |E(G)|$ is as small as possible. Obviously, G is connected. We have the following claims.

Claim 1. *In graph G, each 4^--vertex is not adjacent to any 3^--vertex.*

Proof. Suppose to the contrary that G contains a vertex u of degree $d(u) \leq 4$ which is adjacent to a 3^--vertex v. Assume that $N(u) = \{v\} \cup \{u_i | i \leq 3\}$ and $N(v) = \{u\} \cup \{v_j | j \leq 2\}$. Let $G' = G - uv$. By the minimality of G, there is a k-nsd list total coloring ϕ of G'. We shall now extend the coloring ϕ to G. Since $d(v) \leq 3$, we can erase the color of v and recolor it finally. To guarantee the coloring is proper, we cannot use the colors of edges $uu_i (i \leq 3)$, $vv_j (j \leq 2)$ and vertex u for edge uv, and the