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Two linear-time algorithms for computing the minimum length polygon of a digital contour[☆]

J.-O. Lachaud [a], X. Provençal [a,b,*]

[a] LAMA, UMR CNRS 5127, Université de Savoie, F-73376 Le Bourget du Lac, France

[b] LIRMM, UMR CNRS 5506, Université Montpellier II, 161 rue Ada, F-34000 Montpellier, France

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ABSTRACT

The Minimum Length Polygon (MLP) is an interesting first order approximation of a digital contour. For instance, the convexity of the MLP is characteristic of the digital convexity of the shape, its perimeter is a good estimate of the perimeter of the digitized shape. We present here two novel equivalent definitions of MLP, one arithmetic, one combinatorial, and both definitions lead to two different linear time algorithms to compute them. This paper extends the work presented in Provençal and Lachaud (2009) [26], by detailing the algorithms and providing full proofs. It includes also a comparative experimental evaluation of both algorithms showing that the combinatorial algorithm is about 5 times faster than the other. We also checked the multigrid convergence of the length estimator based on the MLP.

1. Introduction

The minimum length polygon (MLP) or minimum perimeter polygon was proposed long ago for approaching the geometry of a digital contour [25,30]. One of its definitions is to be the polygon of minimum perimeter which stays in the band of 1 pixel-wide centered on the digital contour. It has many interesting properties such as: (i) it is reversible [25]; (ii) it is characteristic of the convexity of the digitized shape and it minimizes the number of inflexion points to represent the contour [30,15]; (iii) it is a good digital length estimator [19,6] and is proven to be multigrid convergent in $O(h)$ for digitization of convex shapes, where h is the grid step (reported in [18,31,32]); (iv) it is also a good tangent estimator; (v) it is the relative convex hull of the digital contour with respect to the outer pixels [30,33] and is therefore exactly the convex hull when the contour is digitally convex.

Several algorithms for computing the MLP have been published. We have already presented the variational definition of the MLP (length minimizer). It can thus be solved by a nonlinear programming method. The initial computation method of [25] was indeed an interactive Newton–Raphson algorithm. Computational complexity is clearly not linear and the solution is not exact. We have also mentioned its set theoretic definition (intersection of relative convex sets). However, except for digital convex shapes, this definition does not lead to a specific algorithm. The MLP may also be seen as a solution to a shortest path query in some well chosen polygon. An adaptation of [14] to digital contour could be implemented in time linear with the size of the contour. It should however be noted that data structures and algorithms involved are complex and difficult to implement. Klette et al. [17] (see also [19,18]) have also proposed an arithmetic algorithm to compute it, but

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* Corresponding author. Fax: +33 4 67 41 85 00.
E-mail addresses: jacques-olivier.lachaud@univ-savoie.fr (J.-O. Lachaud), xavier.provencal@univ-savoie.fr (X. Provençal).

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as it is presented, it does not seem to compute the MLP in all cases. As reported in [9], its edges seem restricted to digital straight segments such that the continued fraction of their slope has a complexity no greater than two.

The MLP is in some sense characteristic of a digital contour. One may expect to find strong related arithmetic and combinatorial properties. This is precisely the purpose of this paper. Furthermore, we show that each of these definitions induces an optimal time integer-only algorithm for computing it. The combinatorial algorithm is particularly simple and elegant, while the arithmetic definition is essential for proving it defines the MLP. These two new definitions give a better understanding of what is the MLP in the digital world. Although other linear-time algorithms exist, the two proposed algorithms are simpler than existing ones. They are thus easier to implement and their constants are better.

The paper is organized as follows. First Section 2 recalls standard definitions. Section 3 gives formally the above-mentioned alternative definitions of the MLP. Section 4 presents how to split uniquely a digital contour into convex, concave and inflexion zones, the arithmetic definition of MLP follows then naturally. Section 5 is devoted to the combinatorial version of MLP. After establishing its equivalence with the arithmetic MLP, we show that our algorithm constructs it in linear time. Section 6 illustrates our results and concludes.

This paper is an extended version of [26]. We provide here full proofs and further examples. We also note that an algorithm for computing the MLP has just been proposed independently by Roussillon et al. (to appear in [28]): it is extremely similar in spirit to our arithmetic algorithm since its computation relies also on maximal segment recognition. However our combinatorial MLP should still be much faster in practice since it does not compute the geometry of segments along the shape.

2. Preliminaries

This section presents the standard definitions that we will used throughout the paper, in order to avoid any ambiguity.

2.1. Polyomino, digital contour, inner and outer polygon

Given some set X in the plane, its *topological interior* will be denoted by X° while its *topological boundary* will be denoted by ∂X.

A *digital square* is a unit closed axis-aligned square in the plane whose center has integer coordinates. A *polyomino* is a set of digital squares in the plane such that its topological boundary is a Jordan curve. It is thus bounded. It is convenient to represent a polyomino as a subset of the digital plane $\mathbb{Z}^2$, which codes the integer coordinates of the centers of its squares, instead of representing it as a subset of the Euclidean plane $\mathbb{R}^2$. When seeing a polyomino as a subset of $\mathbb{R}^2$, we will say *the body of the polyomino*. For instance, the Gauss digitization of a convex subset of the plane is a polyomino iff it is 4-connected. A subset of $\mathbb{Z}^2$, or *digital shape*, is a polyomino iff it is 4-connected and its complement is 4-connected.

In the following, we call *digital contour* the boundary of any polyomino, represented as a sequence of horizontal and vertical steps in the half-integer plane $\left(\mathbb{Z}+\frac{1}{2}\right)\times\left(\mathbb{Z}+\frac{1}{2}\right)$. One can use for instance a Freeman chain to code it as a word over the alphabet $\{0, 1, 2, 3\}$. These words are usually called *contour words*. Again, the *body* of a digital contour is its embedding in $\mathbb{R}^2$ as a polygonal curve. Now, since the body of a digital contour is a Jordan curve, it has one well-defined inner component in $\mathbb{R}^2$, whose closure is exactly the polyomino whose boundary is the digital contour. There is thus a one-to-one map from digital contours to polyominoes, denoted by I.

Let *Sq* be the digital square centered at $(0, 0)$ and let $\oplus$ denotes the Minkowski sum of two sets.

We only deal in this paper with *simple* digital contours (or grid continua in the terminology of [32]). A digital contour C is *simple* if and only if: (i) any digital point of a digital contour C has exactly in its 4-neighborhood two other digital points of C, (ii) the one pixel-wide band $C \oplus Sq$ is an annulus whose topological boundary is composed of two simple closed polygonal lines.

Each of these lines induces a finite simple polygon by Jordan's theorem. The one included in the body of $\mathrm{I}(C)$ is called the *inner polygon* of C and is denoted by $\mathrm{L}_1(C)$. The other one is the *outer polygon* of C and is denoted by $\mathrm{L}_2(C)$. We have thus by definition that $C \oplus Sq = \mathrm{L}_2(C) \setminus \mathrm{L}_1(C)^\circ$. It is easy to check that all digital points on $\partial\mathrm{L}_1(C)$ are in the polyomino $\mathrm{I}(C)$ while all digital points on $\partial\mathrm{L}_2(C)$ are not in the polyomino $\mathrm{I}(C)$. These notions are illustrated on Fig. 1.

2.2. Maximal segments; tangential cover; turns

A *standard digital straight line (DSL)* is some set $\{(x, y) \in \mathbb{Z}^2,\ \mu \le ax - by < \mu + |a| + |b|\}$, where (a, b, μ) are also integers and $\gcd(a, b) = 1$. It is well known that a DSL is a 4-connected simple path in the digital plane, which is the digitization of a Euclidean straight line of slope $\frac{a}{b}$ and shift to origin $-\frac{\mu}{b}$ [27,7]. A *digital straight segment (DSS)* is a 4-connected piece of DSL. Given a digital contour C, a *maximal segment* M is a subset of C that is a DSS and which is no more a DSS when adding any other point of $C \setminus M$.

We recall that the *tangential cover* of a digital contour is the ordered sequence of its maximal segments [12]. In the following, the tangential cover is denoted by $(M_l)_{l=0\cdots m-1}$, where M_l is the l-th maximal segment of the contour. Let us denote by θ_l the slope direction (angle wrt x-axis) of M_l. All indices are taken modulo the number m of maximal segments. Since the directions of two consecutive maximal segments can differ of no greater than π, their variation of direction can