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Some properties for a class of interchange graphs

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ABSTRACT

The Wiener number is the sum of distances between all pairs of vertices of a connected graph. In this paper, we give an explicit algebraic formula for the Wiener number of a class of interchange graphs. Moreover, distance-related properties and cliques of this class of interchange graphs are investigated.

1. Introduction

In this paper, all graphs are finite, undirected, and without loops and multiple edges. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The distance $d_G(u, v)$ between vertices $u, v \in V(G)$ is the number of edges on a shortest path connecting vertices u and v in G. The distance of a vertex $v \in V(G)$, $d_G(v)$, is the sum of all distances between v and all other vertices of G, i.e.,

$$d_G(v) = \sum_{u \in V(G)} d_G(u, v).$$

The Wiener number, which was introduced to define the boiling point of alkane by Harold Wiener [9], is one of the most important topological indices of chemical graphs. It is denoted by $W(G)$ and defined as the sum of distances between all pairs of vertices in G:

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v) = \frac{1}{2} \sum_{v \in V(G)} d_G(v).$$

Wiener number has been widely applied in communications, equipment orientation and cryptography, and so on (cf. Refs. [9,6,5,8,4,7,1,2], and the references therein). Since it deals with distance properties of graphs, computing the Wiener number of a graph is itself an interesting mathematical problem. The research dealing with Wiener number has attracted both chemists and mathematicians, and is still active [6,5,8,4,7,1].

In this paper, we consider interchange graphs. Let $m \geq 2$ and $n \geq 2$ be two positive integers, and let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be two nonnegative integral vectors with $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$. Denote by $U(R, S)$, the set of all $(0, 1)$-matrices

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$A = (a_{ij})_{m\times n}$ with row sum vector R and column sun vector S, i.e.,

$$a_{ij} = 0 \quad \text{or} \quad 1 (i = 1, \ldots, m;\ j = 1, \ldots, n)$$
$$\sum_{j=1}^{n} a_{ij} = r_i \quad (i = 1, \ldots, m)$$
$$\sum_{i=1}^{m} a_{ij} = s_j \quad (j = 1, \ldots, n).$$

Let $A \in U(R, S)$. An interchange of A is a transformation which replaces the 2×2 submatrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of A with the 2×2 submatrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or vice versa. Clearly, an interchange does not alter the row sum vector and the column sum vector of A, and thus replaces a matrix in $U(R, S)$ with another matrix in $U(R, S)$.

Let $G(R, S)$ denote the undirected simple graph, whose vertex set $V(G(R, S)) = U(R, S)$, and where two matrices are joined by an edge if and only if one of them can be obtained from the other by a single interchange [1]. Interchange graphs have provided an interesting research topic. In [1], Brualdi proposed many open problems about interchange graphs. For the work along this line, the readers are referred to [2,10,11,3] and the references therein.

In the following, we confine ourselves to a class of interchange graphs $G(R^*, S^*)$, where $R^* = (r_1, r_2)$ and $S^* = (1, 1, \ldots, 1)$, i.e., $m = 2$ and $n = r_1 + r_2$. The distance properties of $G(R^*, S^*)$ are investigated. An explicit algebraic formula for the Wiener number of $G(R^*, S^*)$ is given. Moreover, it is proved that for any two vertices of $G(R^*, S^*)$ with distance k, there are k^2 internally disjoint paths connecting them.

2. The Wiener number of $G(R^*, S^*)$

Let $A \in V(G(R^*, S^*))$. Note that $R^* = (r_1, r_2)$ and $S^* = (1, \ldots, 1)$. Then $a_{1j} = 1$ (or $a_{1j} = 0$) implies $a_{2j} = 0$ (or $a_{2j} = 1$), $j = 1, 2, \ldots, n$, where $n = r_1 + r_2$. This means that A is uniquely determined by its first row (or second row). Clearly, the first row of A is a vector of dimension n consisting of r_1 ones and $n - r_1$ zeros. In the following, let $H(r, n)$ denote the set of all n dimension vectors of ones and zeros, where $r (\geq 1)$ denotes the number of ones in each n dimension vector of $H(r, n)$. Thus, the number of zeros in each n dimension vector of $H(r, n)$ is $n - r$. Suppose $\tilde{A} \in H(r, n)$. An interchange of $\tilde{A}$ is a transformation which replaces the subvector $(1, 0)$ of $\tilde{A}$ with $(0, 1)$ or vice versa. Let $G(r, n)$ denote the simple undirected graph whose vertex set $V(G(r, n))$ is just $H(r, n)$, and where two n dimension vectors are adjacent if and only if one of them can be obtained from the other by a single interchange. Comparing the definitions of $G(R^*, S^*)$ and $G(r, n)$, we have the following lemmas.

Lemma 2.1. *$G(R^*, S^*)$ is isomorphic to $G(r, n)$: $G(R^*, S^*) \cong G(r, n)$, where $R^* = (r, n - r)$, $S^* = (1, \ldots, 1)$.*

Lemma 2.2. *$G(r, n) \cong G(n - r, n)$.*

By the above two lemmas, we need only to investigate the property of $G(r, n)$ with $r \leq n/2$. Recall that a graph G is said to be k-regular if the degree of each vertex of G is k. The following lemma is straight forward.

Lemma 2.3. *$G(r, n)$ is $r(n - r)$-regular. Moreover,*

$$|V(G(r, n))| = \binom{n}{r}, \qquad |E(G(r, n))| = \frac{1}{2} r(n - r) \binom{n}{r}.$$

Lemma 2.4. *$G(r, n)$ $\left(r \leq \frac{n}{2}\right)$ is a complete graph if and only if $r = 1$.*

Proof. If $r = 1$, it is easy to check that $G(1, n)$ is a complete graph. If $2 \leq r \leq \frac{n}{2}$, there are two different vertices v_i and v_j in $G(r, n)$:

$$v_i = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & \cdots \end{pmatrix}$$
$$v_j = \begin{pmatrix} 0 & 0 & 1 & 1 & \cdots & \cdots \end{pmatrix}.$$

Evidently, v_i cannot be obtained from v_j by exactly one interchange. This means that v_i and v_j are not adjacent in $G(r, n)$. Hence, if $2 \leq r \leq \frac{n}{2}$, $G(r, n)$ is not a complete graph. Therefore, $G(r, n)$ $\left(r \leq \frac{n}{2}\right)$ is a complete graph if and only if $r = 1$. □

Lemma 2.5. *$G(r, n)$ $\left(r \leq \frac{n}{2}\right)$ is bipartite if and only if $n = 2$.*

Proof. When $n = 2$, it is easy to see that $r = 1$ and $G(1, 2) \cong K_2$ is bipartite. Now suppose $n \geq 3$. Then there are three vertices in $G(r, n)$: