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Note

On the complexity of the minimum domination problem restricted by forbidden induced subgraphs of small size

Min Chih Lin, Michel J. Mizrahi *

Instituto de Calculo, Facultad de Ciencias Exactas y Naturales Universidad de Buenos Aires, Buenos Aires, Argentina

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ABSTRACT

We study the computational complexity of the minimum dominating set problem on graphs restricted by forbidden induced subgraphs. We give some dichotomies results for the problem on graphs defined by any combination of forbidden induced subgraphs with at most four vertices, implying either an NP-Hardness proof or a polynomial time algorithm. We also extend the results by showing that dominating set problem remains NP-hard even when the graph has maximum degree three, it is planar and has no induced claw, induced diamond, induced K_4 nor induced cycle of length 4, 5, 7, 8, 9, 10 and 11.

1. Introduction

Dominating set is a fundamental problem of algorithmic graph theory [14] with many applications [7,13,21]. It arises naturally in many areas including mathematics, operations research, logistics, economics, and computer science. A typical application related with the problem is the following: Assume you have a representation of a city given by a grid, where intersection points are the corners, and you want to put cameras on it to observe the entire grid (city). The goal is to choose a set of points in the plane in order to *observe* the remaining points. The grid can be represented as a graph where corners are vertices and adjacent corners are joined by edges. Since one can be interested in the observation of certain regions of the city, it makes sense to use as representation an induced graph of the grid. These graphs are a subclass of K_3-free, which is one of the classes we analyze.

The problem remains NP-hard in many restricted graph families such as planar [18], chordal [3], split and bipartite [1], planar graphs of maximum degree 3 [11], among others. On the other hand, the problem has efficient algorithms for classes such as interval and circular-arc graphs [5], AT-free graphs [16], and strongly chordal graphs [10]. Many graph classes can be defined by forbidding certain induced subgraphs. For instance, split graphs, interval graphs, bipartite graphs, permutation graphs, among others can be defined as $\mathcal{F}$-free graphs, for some graph family $\mathcal{F}$. Several of them are interesting from the geometric point of view. In this paper we present a systematic analysis of the complexity for the minimum dominating set problem on $\mathcal{F}$-free graphs, where $\mathcal{F}$ is a family of graphs of order at most four. For unary family $\mathcal{F} = \{F\}$, usually we denote F-Free instead of $\mathcal{F}$-free.

The next section gives the notations used throughout this paper. In the third section we show previous results that will be used for simplifying the analysis, along with some remarks that were not mentioned previously but are needed for our work. At the end of this section we present the current known information, and also give a graph representation of the results as a help for visualization. In the following section we prove the complexity of the domination problem for several graph classes in order to obtain the desired results. Finally we give the conclusions of this paper.

* Corresponding author.
E-mail addresses: oscarlin@dc.uba.ar (M.C. Lin), michel.mizrahi@gmail.com (M.J. Mizrahi).

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2. Notations

Let $G = (V, E)$ be an undirected graph where $V(G)$ and $E(G)$ denote the vertex set and the edge set respectively. Throughout the paper, $n_G = |V(G)|$ and $m_G = |E(G)|$ denote the numbers of vertices and edges of the graph G, respectively. Denote by $N_G(v)$ the subset of vertices adjacent to v, and let $N_G[v] = N_G(v) \cup \{v\}$. Set $N_G(v)$ is called the *neighborhood* of v, while $N_G[v]$ is the *closed neighborhood* of v. Let $S \subseteq V(G)$, we denote the *neighborhood* of S as $N_G(S) = \bigcup_{v \in S} N_G(v) \setminus S$ and the *closed neighborhood* of S as $N_G[S] = N_G(S) \cup S$. The *degree* of v is $d_G(v) = |N_G(v)|$. By $\Delta(G)$ we denote the maximum vertex degree in G. When there is no ambiguity, we may omit the subscripts from n, m, N and d. We say that u is *universal* when $N[u] = V(G)$. Say that w is *dominated* by vertex v if $N[w] \subseteq N[v]$. A subset S of V dominates another subset T of V if $T \subseteq N[S]$.

As usual, C_n and P_n denote the chordless cycle and the chordless path on n vertices, respectively. A *dominating set* of G is a set $W \subseteq V(G)$ such that every vertex in $V(G) \setminus W$ is adjacent to some vertex of W. The size of a minimum dominating set in a graph G is called the domination number of G and is denoted as $\gamma(G)$. An induced subgraph H of G is said *dominating H* if $V(H)$ is a dominating set of G. Clearly, if there is a dominating H then $\gamma(G) \leq |V(H)|$.

The graph K_q is the complete graph of q vertices, and the graph tK_q consists of the disjoint union of t copies of the graph K_q.

Subdivision of an edge is the operation of creation of a new vertex on the edge. When a polynomial-time algorithm has been shown for a problem, we say that the problem is in P. Whenever the problem is in the complexity class NP-Complete we say that the problem is *NPC*.

We assume G is connected, since domination problem can be solved independently in each connected component.

3. Previous results

The set of graphs of order three are: P_3, K_3, $3K_1$ and co-P_3. Since P_3-free is a subclass of P_4-free, which is a subclass of permutation graphs, then by applying [6], we deduce that domination problem restricted to P_3-free graphs is in P. In addition, P_3-free graphs are also known as cluster graphs which are disjoint unions of complete graphs and there is a trivial linear-time algorithm to solve the minimum dominating set problem in this class of graphs. From [1] it is known that domination problem restricted to bipartite graphs is in *NPC*, hence for K_3-free graphs the problem is also in *NPC* and from [16] we can conclude that domination restricted to co-paw-free graphs or $3K_1$-free graphs is in P since both are subclasses of AT-free graphs.

Lemma 3.1 ([1]). *Domination problem restricted to split graphs is in NPC.*

Corollary 3.2. *Since $(2K_2, C_4)$-free graphs is a superclass of split graphs, then domination problem restricted to this class is also in NPC.*

Lemma 3.3 ([23]). *Domination Problem restricted to (K_p, P_5)-free graphs for fixed p is in P.*

Corollary 3.4. *Domination problem restricted to $(2K_2, K_4)$-free graphs is in P.*

Lemma 3.5 ([2]). *The clique-width of (claw, co-claw)-free graphs and (claw, paw)-free graphs is bounded.*

Lemma 3.6 ([9]). *The clique-width of $(K_2 \cup \text{claw}, K_3)$-free graphs is bounded.*

Lemma 3.7 ([8]). *Domination problem is in P for graph classes with bounded clique-width.*

Corollary 3.8. *Domination problem restricted to (claw, co-claw)-free graphs or (claw, paw)-free or $(K_2 \cup \text{claw}, K_3)$-free graphs is in P.*

We add some remarks for easy results for which we could not find any references:

Remark 3.9. Note that (claw, K_3)-free $\subseteq$ $(K_2 \cup \text{claw}, K_3)$-free graphs. Hence the domination problem on (claw, K_3)-free graph class is in P.

Remark 3.10. The minimum dominating set problem restricted to $4K_1$-free graphs is in P.

Proof. Given a $4K_1$-free graph G, any maximal independent set I of G has at most 3 vertices. It is well-known that every maximal independent set is a dominating set, then there is at least one dominating set of size at most 3. Hence, the minimum dominating set for G can be solved by checking for each possible subset of at most three vertices if it is a dominating set. This can be done in $O(n^3)$. □

Remark 3.11. The minimum dominating set problem restricted to co-diamond-free graphs is in P.

Proof. Given a co-diamond-free graph G, if G has no edge then $\gamma(G) = n$. Otherwise, let $e = v_1v_2$ be an arbitrary edge. If $\{v_1, v_2\}$ is not a dominating set then $\exists\, w$ such that $\{v_1, v_2\} \cap N(w) = \emptyset$. Clearly, $\{v_1, v_2, w\}$ is a dominating set because if exists some vertex $x \in V(G)$ such that is not adjacent to any of them, then $\{v_1, v_2, w, x\}$ induces a co-diamond which is a contradiction. □