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Note

Inapproximability results related to monophonic convexity

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ABSTRACT

In 2010, it was proved that the interval number and the convexity number on the monophonic convexity are NP-hard on general graphs (Dourado et al., 2010). In this paper, we extend this results on the monophonic convexity. We prove that deciding if the interval number is at most 2 and deciding if the percolation time is at most 1 are NP-complete problems even in bipartite graphs. We also prove that the convexity number is as hard to approximate as the maximum clique problem. Finally, we obtain polynomial time algorithms to determine the convexity number on hereditary graph classes such that the computation of the clique number is polynomial time solvable (as perfect graphs and planar graphs).

1. Introduction

The fundamental problem of *Distance Geometry* is finding a set of points in some given geometric space (like the euclidean space or a graph) by using the distances between some pairs of such points. From [12], "*Distance Geometry has strong connections to matrix analysis, semidefinite programming, convex geometry and graph rigidity*". In this paper, we focus on convex geometries on graphs.

Convexity spaces have been considered in different branches of mathematics. The study of convexities applied to graphs has started more recently, about 50 years ago. Abstract convexity parameters, when considered on graph convexities, give rise to interesting graph parameters. In particular, complexity aspects related to the computation of these parameters have been the main goal of various recent papers.

The computation of convexity parameters for a graph depends on the particular convexity being considered. Among the existing graph convexities, we can mention the *geodesic convexity* and the *monophonic convexity*, whose convex sets are based on minimum paths and induced paths of the graph, respectively. In some graph classes, like the *distance-hereditary graphs* (which are the graphs such that every induced path is a minimum path), the geodesic convexity and the monophonic convexity are equivalent.

In this paper, we focus on the monophonic graph convexity. Let G be a simple finite graph with vertex set $V(G)$ and let $S \subseteq V(G)$. The *interval* $I(S)$ is the set S plus every vertex outside S belonging to some induced path between two vertices in S. If $I(S) = V(G)$, we say that S is an *interval set*. The *interval number* $in(G)$ is the size of the minimum interval set. In some papers, interval set is called *monophonic set* and interval number is called *monophonic number* [15].

We say that S is *convex* if $I(S) = S$. The *convex hull* of S, denoted by $hull(S)$, is the minimum convex set which contains S. If $hull(S) = V(G)$, we say that S is a *hull set*. The *hull number* $h(G)$ is the size of the minimum hull set of G. The *convexity number* $cx(G)$ is the size of the maximum convex set distinct from $V(G)$.

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Let $t(S)$ be the minimum k such that $I^k(S) = I^{k+1}(S)$, where I^k is the k-th iterate of the interval function $I(\cdot)$. The *percolation time* $t(G)$ is the maximum $t(S)$ among all hull sets S. The percolation time and some variants were already investigated in other different convexities as the P_3-convexity [2,3,1,13] and the *geodesic convexity* [6,10].

In [8], P. Duchet proved that the Carathéodory number is 1 for complete graphs and 2 for other graphs, the Helly number is the clique number and the Radon number is the clique number plus one except possibly for triangle-free graphs, where it is at most 4.

In 2010, it was proved that the interval number and the convexity number are NP-hard on general graphs [7]. Interestingly, they obtained a polynomial time algorithm to compute the hull number of a graph.

In this paper, we extend some of these results. In Section 2, we prove that deciding if the interval number is at most 2 is NP-complete even in bipartite graphs. In Section 3, we prove that deciding if the percolation time is at most 1 is NP-complete even in bipartite graphs. In Section 4, we prove that the convexity number is as hard to approximate as the maximum clique problem, that is, for every $\varepsilon > 0$, there is no polynomial time $O(n^{1-\varepsilon})$-approximation algorithm for the convexity number, unless P = NP.

In Section 4, we also obtain polynomial time algorithms to determine the convexity number on hereditary graph classes such that the computation of the clique number is polynomial time solvable (as perfect graphs and planar graphs). We say that a graph class is hereditary if, for every induced subgraph H of a graph G in this graph class, we have that H is also in this graph class. In Section 5, we present some conclusions and applications to network localization problems.

2. The interval number of the monophonic convexity

In 2010, it was proved the following theorem [14], which is very useful in this section.

Theorem 2.1 (*[14]*). *Given a bipartite graph G and three distinct vertices x, y, z, deciding whether there is an induced path from x to y passing through z is NP-complete.*

As a direct consequence, we have that, given a connected bipartite graph G and distinct vertices x, y, z, deciding if $z \in I(\{x, y\})$ is NP-complete. In other words, the problem of determining the interval of a set X is NP-hard, even if $|X| = 2$ and the graph is bipartite.

The following theorem proves that deciding if a set S of vertices is an interval set is NP-complete, even if the graph is bipartite and S has at most 2 elements. The theorem also proves that deciding whether $in(G) \leq 2$ is NP-complete even in bipartite graphs.

Theorem 2.2. *Given a connected bipartite graph G and $S \subseteq V(G)$ with $|S| = 2$, the following problems are NP-complete:* (a) *deciding if S is an interval set;* (b) *deciding if $in(G) \leq 2$.*

Proof. A certificate that the first problem belongs to NP is a set of at most $|V(G)| - |S|$ induced paths, each one beginning and finishing in distinct vertices of S, such that every vertex of $V(G) \setminus S$ appears in at least one of these paths. A certificate that the second problem belongs to NP is a set S with at most 2 vertices and a set of at most $|V(G)| - |S|$ induced paths between two vertices of S, such that every vertex of $V(G) \setminus S$ belongs to at least one of these paths.

We first prove the NP-completeness of the first problem. We describe a reduction from the problem of deciding if $z \in I(\{x, y\})$. Let H be a bipartite graph with bipartition (A, B) and let x, y, z be three distinct vertices of H. We can assume that x is not neighbor of z (otherwise, subdivide the edge xz twice). Analogously, we can assume that y is not neighbor of z. Without loss of generality, suppose that $z \in B$. Define a bipartite graph G by adding to H 10 new vertices a_i, b_i, c_i, d_i, e_i ($i \in \{1, 2\}$) such that a_1 and a_2 are adjacent to all vertices in $B \setminus \{z\}$, and b_1 and b_2 are adjacent to all vertices in A. Also include the edges $c_id_i, d_ie_i, e_ib_i, b_ia_i$ ($i \in \{1, 2\}$). Finally, if $x \in A$, include the edge xd_1; otherwise, include the edge xe_1. If $y \in A$, include the edge yd_2; otherwise, include the edge ye_2. Clearly G is bipartite with bipartition $(A \cup \{a_i, c_i, e_i : i \in \{1, 2\}\}, B \cup \{b_i, d_i : i \in \{1, 2\}\})$. Set $S = \{c_1, c_2\}$.

We have to show that $z \in I(\{x, y\})$ in H if and only if S is an interval set of G. Let $h_1 = d_1$, if xd_1 is an edge, and $h_1 = e_1$, otherwise. Let $h_2 = d_2$, if yd_2 is an edge, and $h_2 = e_2$, otherwise. Notice that $x \in I(\{c_1, c_2\})$, since there is either the induced path $c_1d_1xb_2e_2d_2c_2$ or the induced path $c_1d_1e_1xa_2b_2e_2d_2c_2$. Analogously, $y \in I(\{c_1, c_2\})$. Notice that $A \subseteq I(\{c_1, c_2\})$, since, for every vertex $v \in A \setminus \{x, y\}$, there is the induced path $c_1d_1e_1b_1vb_2e_2d_2c_2$. Also notice that $B \setminus \{z\} \subseteq I(\{c_1, c_2\})$, since, for every vertex $v \in B \setminus \{x, y, z\}$, there is the induced path $c_1d_1e_1b_1a_1va_2b_2e_2d_2c_2$. Thus $I(\{c_1, c_2\}) \supseteq V(G) \setminus \{z\}$.

Assume that $z \in I(\{x, y\})$ in H. Since every induced path of H is also an induced path of G, then $z \in I(\{x, y\})$ in G. This implies the existence of an induced path $c_1 - h_1x - z - yh_2 - c_2$. Then $z \in I(\{c_1, c_2\})$ and consequently $I(\{c_1, c_2\}) = V(G)$.

Now assume that $I(\{c_1, c_2\}) = V(G)$. This implies that $z \in I(\{c_1, c_2\})$ and there is an induced path P from c_1 to c_2 passing through z. We claim that P is of the type $c_1 - h_1x - z - yh_2 - c_2$ and, thus, P contains an induced path from x to y passing through z.

Note that there is no induced path from c_1 to c_2 passing through z and b_1, since every induced path containing z must have two neighbors of z, which are in A and are both adjacent to b_1. The same for b_2. Also note that there is no induced path passing through z and a_1, since every induced path containing z and not containing b_1 or b_2 must contain two vertices of B (recall that xz and yz are not edges), which are adjacent to a_1. The same for a_2.