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On the complexity of the flow coloring problem

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ABSTRACT

Motivated by bandwidth allocation under interference constraints in radio networks, we define and investigate an optimization problem that combines the classical flow and edge coloring problems in graphs. Let $G = (V, E)$ be a graph with a demand function $b : V \to \mathbb{Z}_+$ and a *gateway* $g \in V \setminus V_s$, where $V_s = \{v \in V : b(v) > 0\}$ is the set of *source nodes*. A flow ϕ_u of a source node u is a multiset of $b(u)$ paths in G from u to g. A flow ϕ on G is a set with one flow for each source node. Every flow ϕ defines a multigraph G_ϕ with vertex set V and all edges in the paths in ϕ. A *distance-d edge coloring* of a flow ϕ is an edge coloring of G_ϕ such that two edges with the same color are at distance at least d in G. The *distance-d flow coloring problem* (FCP_d) is the problem of obtaining a flow ϕ on G with a distance-d edge coloring where the number of used colors is minimum. For any fixed $d \geq 3$, we prove that FCP_d is NP-hard even on a bipartite graph with just one source node. For $d = 2$, we also prove NP-hardness on a bipartite graph with multiples sources. For $d = 1$, we show that the problem is polynomial in 3-connected graphs and bipartite graphs. Finally, we show that a list version of the problem is inapproximable in polynomial time by a factor of $O(\log n)$ even on n-vertex paths, for any $d \geq 1$.

1. Introduction

Let us consider a scenario in a radio network where source nodes have to repeatedly send a certain amount of information to a gateway through the network (possibly using intermediate nodes to pass on the information). Some links cannot transmit simultaneously due to interference constraints. To overcome this difficulty, the transmission is organized by rounds, which is defined by the links transmitting at the same time. A sequence of rounds is then successively repeated to keep the required flow of information from the sources to the gateway. The objective is to find a minimum multiset of rounds to carry over the traffic demand. Once the information is continuously transmitted over time, the focus is the system throughput whereas the length of the paths used to send flow from a source node to the gateway is not an issue in this case. This is the scenario modeled by the Round Weighting Problem—RWP [11,8].

Such an application combines two classical problems in graphs. A flow has to be sent through a graph, and a coloring of the edges carrying flow has to be determined (each color defines a round). While the flow problem only depends on the network structure and the demand, the coloring problem depends on the interference model considered. In general, radio

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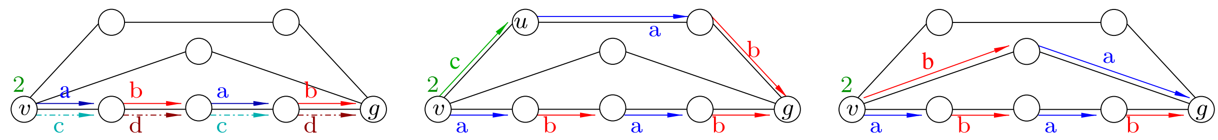

(a) $w(G) = 4$. (b) $w(G) = 3$. (c) $w(G) = \chi'_{\Phi,d}(G) = 2$.

Fig. 1. Feasible solutions for FCP_1.

transmissions are subjected to different interference constraints (see for instance Bermond et al. [2]), but a central factor is the distance between the connections. An interference model based on distance is used for example in Barrett et al. [1].

Inspired by this application, the idea of combining flow and coloring was formalized by Campêlo et al. [4], where the term *flow coloring* was adopted. Instead of rounds, color classes are used to cover the edges carrying flow. Counterparts of classical coloring parameters appear naturally. It is the case of the flow chromatic index, where the color classes are matchings [4]. Here, we consider more general color classes, based on the distances between the edges.

Let $G = (V, E)$ be a simple, connected graph. The distance between two vertices $u, v \in V$, to be denoted $dist(u, v)$, is the length of a minimum path between u and v in G, where the length of a path is the number of edges it comprises. The distance between two edges is the minimum distance between their end vertices. Given an integer $d \geq 1$, a *distance-d edge set* is a set of edges which are pairwise at distance at least d in G. Let $\mathcal{M}_d$ be the set of all distance-d edge sets of G. In particular, $\mathcal{M}_1$ is the set of all matchings of G.

In the connected graph $G = (V, E)$, let $g \in V$ be a special node to be called *gateway* or *destination node*, and consider a function $b : V \to \mathbb{Z}_+$ which associates to every node u an integer demand $b(u)$ to be sent to g through G. We assume that $b(g) = 0$ and call *source node* a node u with $b(u) > 0$. Let $b(G)$ denote the total demand, i.e. $b(G) = \sum_{u \in V} b(u)$.

A *flow* ϕ_u *of a source node* u is a multiset of $b(u)$ paths from u to the gateway g (these paths are not necessarily distinct nor disjoint). A flow ϕ on G is a set containing one flow ϕ_u for each source node u. A flow is therefore a function $\phi : E \to \mathbb{Z}_+$, where $\phi(e)$ is the amount of flow passing through $e \in E$. Let Φ stand for the set of all possible flow functions in G.

The *distance-d flow coloring problem* (FCP_d) consists in determining a weight function $w : \mathcal{M}_d \to \mathbb{Z}_+$ that minimizes $w(G) := \sum_{M \in \mathcal{M}_d} w(M)$ and covers a flow in G, that is, for some $\phi^\star \in \Phi$,

$$\sum_{M \in \mathcal{M}_d : e \in M} w(M) \geq \phi^\star(e) \quad \forall e \in E.$$

In the aforementioned application, M models a round and $w(M)$ is the number of times it is used. The flow $\phi^\star$ is the one chosen to route the demand to the gateway.

In this work, we study the computational complexity of the problem FCP_d. We prove NP-hardness even on bipartite graphs with one source (for $d \geq 3$) and multiple sources (for $d = 2$). On the other hand, for $d = 1$, we prove that the problem is polynomially solvable in bipartite graphs. Recall that k-connectivity in a graph means k vertex-disjoint paths between any pair of non-adjacent vertices [13]. In the context of the application, such a property relates to the survivability of the network. Having alternative paths from a source to the gateway may guarantee better tolerance to faults, inoperative links, etc. By its turn, analyzing the problem in bipartite graphs are twofold relevant. From the application point of view, this graph class includes trees and grids, which are basic network configurations. From the theoretical point of view, it is a basic graph class to consider when analyzing the computational complexity of a problem in graphs.

2. The distance-*d* flow coloring problem—*FCP$_d$*

We start this section by giving an equivalent definition of the distance-d flow coloring problem, which will be useful in the remaining of the text. Notice that every flow $\phi \in \Phi$ defines a multigraph $G_\phi = (V, \phi)$ obtained from G with $\phi(e)$ copies of each edge $e \in E$ (an edge $e \in E$ with $\phi(e) = 0$ is removed). We define a *distance-d edge coloring of a flow* ϕ as an edge coloring of G_ϕ such that edges with the same color are at distance at least d in the original graph G. Notice that each color class is a distance-d edge set (i.e. an element of $\mathcal{M}_d$), and the coloring defines a weight function $w : \mathcal{M}_d \to \mathbb{Z}_+$, being $w(M)$ the number of times color class M is used. Therefore, the *distance-d flow coloring problem* (FCP_d) consists in obtaining a flow $\phi^\star \in \Phi$ with a distance-d edge coloring using the least number of colors. The minimum number of used colors will be called the *distance-d flow chromatic index* $\chi'_{\Phi,d}(G)$.

Fig. 1 illustrates an example for $d = 1$. There is a unique source node v with demand $b(v) = 2$ to be sent to the gateway g. The arrows describe the flow. The letter associated with each arrow corresponds to its color. In Fig. 1(a), the flow is sent through a path. Two elements (matchings) of $\mathcal{M}_d$ are used, each of them with weight 2, thus giving a total of 4 colors. In general, observe that any path in G with size $p \leq d + 1$ needs at least p colors. Figs. 1(b) and (c) present other feasible solutions with less colors. Now the flow is sent through two vertex-disjoint paths. In particular, when these paths form an even cycle (Fig. 1(c)), we get the optimal solution.

In other terms, the distance-d flow coloring problem implicitly considers all the distance-d edge colorings of all multigraphs that have a subgraph of G as the underlying graph and whose edge multiplicity can be given by a flow on G. The solution corresponds to one of those multigraphs with a distance-d edge coloring using the minimum number of colors.