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Facial edge ranking of plane graphs

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ABSTRACT

A facial edge k-ranking of a plane graph G is a labeling of its edges with integers $1, \ldots, k$ such that every facial trail connecting two edges with the same label contains an edge with a greater label. The smallest integer k such that G has a facial edge k-ranking is denoted by $\chi'_{fr}(G)$. We prove that $\chi'_{fr}(G) = O(\log \Delta^*)$ for 3-edge-connected plane graphs, where Δ^* is the maximum face size of G.

1. Introduction

Consider a simple graph G. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. An *edge k-ranking* of G is a function $c : E(G) \to \{1, \ldots, k\}$ such that each path connecting two edges x, y satisfying $c(x) = c(y)$ contains an edge z such that $c(z) > c(x)$. The smallest integer k such that G admits an edge k-ranking is denoted by $\chi'_r(G)$.

The edge ranking problem is to find $\chi'_r(G)$ of given graph G. This problem has applications in the parallel assembly of modular products from their components [4,8] or in the parallel database query processing [5,13]. The edge ranking problem is known to be NP-hard for general graphs [10]. Some polynomial time algorithms have been developed for a few special graphs, e.g., trees [3], 2-connected outerplanar graphs [14], complete graphs [1].

In this paper we are investigating a relaxation of the edge ranking problem for the case of plane graphs, where the constraints are given by faces. There are several very recent papers that study different types of colorings/labelings of plane graphs where constraints on colorings/labelings are given by faces (see [9,11,12,16] and references therein). These papers also motivated us to introduce our new problem.

A graph which can be embedded in the plane is called planar graph; a fixed embedding of a planar graph is called *plane graph*. Let G be a connected plane graph. Let f be a face of G of size k having boundary walk $v_0, e_0, v_1, \ldots, v_{k-1}, e_{k-1}, v_k = v_0$ with $v_i \in V(G), e_i \in E(G)$ and $e_i = v_i v_{i+1}$ for every $i = 0, 1, \ldots, k-1$ (for the definition of the boundary walk see [7, p. 101]). A *facial trail* of f is any trail of the form $v_m, e_m, v_{m+1}, \ldots, v_{n-1}, e_{n-1}, v_n$ (indices modulo k), where v_i and e_i are vertices and edges of the boundary walk of f. A facial trail in G is any facial trail of some face f.

In this paper we consider the facial edge ranking problem of plane graphs which can be considered as a relaxation of the edge ranking problem. We focus on facial trails of plane graphs. A *facial edge k-ranking* of a plane graph G is a labeling of its edges with integers $1, \ldots, k$ such that every facial trail connecting two edges with the same label contains an edge with a greater label. The smallest integer k such that G has a facial edge k-ranking is denoted by $\chi'_{fr}(G)$. The number $\chi'_{fr}(G)$

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is called *facial ranking index* of G. Observe that this labeling need not be proper in a usual way. We require only that *face-adjacent* edges (consecutive edges of a facial trail of some face) must receive different labels. On the other hand these types of labelings coincide in the class of paths and cycles.

For plane triangulations the facial edge ranking problem is equivalent to the four color problem, see e.g. the book of Saaty and Kainen [17]. From the Four Color Theorem the following result follows, see [17, p. 103].

Theorem 1. *The edges of any plane triangulation T can be colored with 3 colors so that the edges bounding every face are colored distinctly, i.e.*

$$\chi'_{fr}(T) = 3.$$

Shannon [19] proved that every multigraph G with maximum degree Δ has a proper edge coloring with at most $\frac{3}{2}\Delta$ colors. This result can be reformulated for the family of plane graphs in the following way.

Theorem 2. *Let G be a 2-edge-connected plane graph with maximum face size Δ^*. Then the edges of G can be colored with at most $\frac{3}{2}\Delta^*$ colors in such a way that the edges bounding every face of G are colored distinctly.*

Corollary 1. *If G is a 2-edge-connected plane graph with maximum face size Δ^*, then*

$$\chi'_{fr}(G) \le \frac{3}{2}\Delta^*.$$

Vizing [21,20] proved that simple planar graphs with maximum degree at least eight have the chromatic index (edge chromatic number) equal to their maximum degree. He conjectured the same if the maximum degree is either seven or six. The first part of this conjecture was proved by Sanders and Zhao [18]. Note that (also by Vizing) every graph with maximum degree Δ has the chromatic index equal to Δ or $\Delta + 1$. These results of Sanders and Zhao and of Vizing can be reformulated in the following way.

Theorem 3. *Let G be a 3-edge-connected plane graph with maximum face size $\Delta^* \ge 7$. Then the edges of G can be colored with Δ^* colors in such a way that the edges bounding every face of G are colored distinctly.*

Corollary 2. *If G is a 3-edge-connected plane graph with maximum face size $\Delta^* \ge 7$, then*

$$\chi'_{fr}(G) \le \Delta^*.$$

Bruoth and Horňák [2] determined the value of $\chi'_r(C_n)$ for any cycle C_n.

Theorem 4 ([2]). *Let C_n be a cycle on $n \ge 3$ edges. Then*

$$\chi'_r(C_n) = \lfloor \log_2(n-1) \rfloor + 2.$$

Corollary 3. *If G is a 2-edge-connected plane graph with maximum face size Δ^*, then*

$$\chi'_{fr}(G) \ge \left\lfloor \log_2(\Delta^* - 1) \right\rfloor + 2.$$

In this paper, we show that $\chi'_{fr}(G) = O(\log \Delta^*)$ for every 3-edge-connected plane graph G, where Δ^* is the maximum face size of G.

2. Results

Note that the facial ranking index depends on the embedding of the graph G. For example, the graph depicted in Fig. 1 with the embedding on the left has no facial edge 4-ranking (since the boundary of the outer face is a 9-cycle and from Theorem 4 it follows that $\chi'_r(C_9) = 5$); whereas with the embedding on the right, it has a facial edge 4-ranking.

The *dual* G^* of a plane graph G can be obtained as follows: Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G (an edge separates the faces incident with it).

2.1. 3-connected plane graphs

We write $v \in f$ if a vertex v is incident with a face f. Two distinct faces f and g *touch* each other, if there is a vertex v such that $v \in f$ and $v \in g$.