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Triangulations and equality in the domination chain

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ABSTRACT

In this paper we give a characterization of triangulations where all minimal dominating sets have the same cardinality and a characterization of triangulations where all six domination parameters are the same.

1. Introduction

In this paper, all graphs are simple and finite. A graph is *planar* if it can be embedded in the plane, and a graph is *plane* if it is drawn with such an embedding. A graph is called a *triangulation* if it is plane and every face is a triangle. An *induced subgraph* H of G is a graph whose vertex set $V(H) = S$ is a subset of $V(G)$ and contains all the edges between pairs of vertices in $V(H)$ that are contained in G. Such a graph is denoted $H = G[S]$. In this paper, we adopt the convention that an induced subgraph of a plane graph G inherits the corresponding embedding from G.

The *open neighbourhood*, denoted $N(v)$, of a vertex v is the set of vertices adjacent to v, and the *closed neighbourhood*, denoted $N[v]$, is $N(v) \cup \{v\}$. For $S \subseteq V(G)$, $N(S) = \bigcup_{x\in S} N(x)$ and $N[S] = S \cup N(S)$.

A subset I of $V(G)$ is an *independent set* if no two vertices in I are adjacent. A subset D of $V(G)$ is a *dominating set* if $N[D] = V(G)$. If I is a subset of $V(G)$, and $x \in I$, the set of *I-private neighbours* of x, denoted $PN(x, I)$, is $N[x] - N[I - \{x\}]$. If $x \in PN(x, I)$, then x is called an *I-self private neighbour* and if $y \neq x$ and $y \in PN(x, I)$, then y is called an *I-external private neighbour* (of x). A subset I of $V(G)$ is an *irredundant set* if for each $x \in I$, $PN(x, I) \neq \emptyset$, i.e., every vertex in I has an I-private neighbour. We say a vertex $y \notin I$ *annihilates* a vertex $x \in I$ if $PN(x, I) \subseteq N[y]$, which implies $I \cup \{y\}$ is not an irredundant set.

For a graph G, its *lower independence number*, denoted $i(G)$, is the minimum cardinality of a maximal independent set of G, and its *upper independence number*, denoted $\alpha(G)$, is the maximum cardinality of an independent set of G. Similarly, its *lower domination number*, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G, its *upper domination number*, denoted $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G, its *lower irredundance number*, denoted $ir(G)$, is the minimum cardinality of a maximal irredundant set of G, and its *upper irredundance number*, denoted $IR(G)$, is the maximum cardinality of an irredundant set of G. It is well known that every maximal independent set is a minimal dominating set and every minimal dominating set is a maximal irredundant set; this implies a relation of inequalities between the parameters, widely known as the *domination chain*:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G).$$

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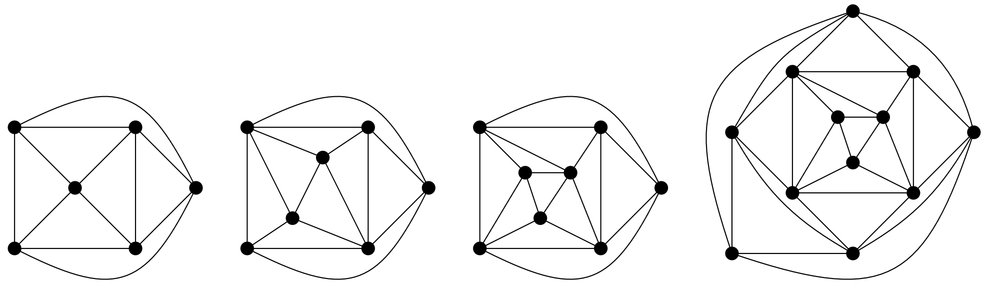

(a) R_6. (b) R_7. (c) R_8. (d) R_{12}.

Fig. 2.1. The four 4-connected well-covered triangulations.

Equality in the domination chain has been studied extensively. Cockayne and Mynhardt [2] characterize all possible 6-tuples for which there exists a graph whose parameters of the domination chain are the values of the 6-tuple. A graph G is called *well-covered* if each maximal independent set of G has the same cardinality, equivalently $i(G) = \alpha(G)$. The study of well-covered graphs was first proposed by Plummer [11], and the reader is also referred to the surveys of Hartnell [10] and Plummer [12] for a complete introduction to well-covered graphs. Chvátal and Slater [1] and Sankaranarayana and Stewart [13] have independently shown that the recognition problem of well-covered graphs is co-NP-complete. A graph is called *well-dominated* if each minimal dominating set of G has the same cardinality, equivalently $\gamma(G) = \Gamma(G)$, and a graph is called *well-irredundant* if each maximal irredundant set of G has the same cardinality, equivalently $ir(G) = IR(G)$. Well-dominated graphs were first studied by Finbow et al. [3], and the study of well-irredundant graphs was introduced by Topp and Vestergaard [14]. The complexity of recognizing well-dominated and well-covered graphs is open.

In this paper, we extend the characterization of Finbow et al. [6–9] of well-covered triangulations to characterize the well-dominated triangulations and the well-irredundant triangulations. A triangulation is also a maximal (with respect to edge addition) planar graph and therefore have received much attention. For example, an early observation in showing every planar graph was four colourable, was that the result would follow if one could show every triangulation was four colourable. Triangulations were a natural class to study in the context of well-covered, well-dominated and well-irredundant graphs as have many small cycles and there are several characterizations of classes of well-covered graphs where small cycles are forbidden (see, for example [4,5,3,14]).

2. Well-covered triangulations

In a sequence of four papers, Finbow et al. [6–9] characterized the well-covered triangulations. Clearly, the only triangulation which is exactly 2-connected is K_3, which is trivially well-covered. In the first paper, the authors showed that there are no 5-connected well-covered triangulations. In the next two papers, the authors showed that there are exactly four 4-connected well-covered triangulations, namely the graphs R_6, R_7, R_8, and R_{12} shown in Fig. 2.1. In the final paper, the characterization of well-covered triangulations was completed by characterizing those which are exactly 3-connected. We extend the results below to characterize both the well-dominated triangulations and the well-irredundant triangulations. We first introduce the definitions necessary for the characterization.

Definition 2.1 (*[9]*)**.** Let F and G be graphs and suppose that abc and rst are triangular faces in disjoint copies of F and G respectively. Then an *O-join* of F and G at faces abc and rst is a new graph H such that $V(H) = V(F) \cup V(G)$ and $E(H) = E(F) \cup E(G) \cup \{ar, as, bs, bt, ct, cr\}$. If H is the O-join of graphs F and G and the faces at which the O-join is made are irrelevant, we will denote H by $F \bigcirc G$.

Definition 2.2 (*[9]*)**.** If G is a well-covered graph and abc is a triangular face of G, then abc is called a *YES-face* if each of $G - a - b$, $G - a - c$, and $G - b - c$ is also well-covered. A triangular face of G that is not a YES-face is called a *NO-face*.

Lemma 2.3 (*[9]*)**.** *In R_8, labelled as shown in Fig. 2.2, triangles abc, abf, edh, and gdh are YES-faces, while all other triangular faces are NO-faces. Furthermore, for each pair of YES-faces S and T in R_8, there exists another embedding σ of R_8 such that $\sigma(S) = T$.*

We note that when F or G or both are isomorphic to R_8, and the faces at which the O-join is made are irrelevant, provided they are YES-faces, we continue to denote the O-join of F and G with $F \bigcirc G$.

Definition 2.4 (*[9]*)**.** *The K_4-family $\mathcal{K}$* is the collection of all plane graphs consisting of a set S of vertex-disjoint K_4s, together with arbitrary edges added so as to form a plane triangulation where each element of S has at least three faces in common with the resulting graph.

Definition 2.5 (*[9]*)**.** The *extended K_4-family*, denoted $\mathcal{K}^+$, is