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On set expansion problems and the small set expansion conjecture[☆]

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ABSTRACT

We study two problems related to the Small Set Expansion Conjecture (Raghavendra and Steurer, 2010): the Maximum weight m'-edge cover (MWEC) problem and the Fixed cost minimum edge cover (FCEC) problem. In the MWEC problem, we are given an undirected simple graph $G = (V, E)$ with integral vertex weights. The goal is to select a set $U \subseteq V$ of maximum weight so that the number of edges with at least one endpoint in U is at most m'. Goldschmidt and Hochbaum (1997) show that the problem is NP-hard and they give a 3-approximation algorithm for the problem. The approximation guarantee was improved to $2 + \epsilon$, for any fixed $\epsilon > 0$ (Liang, 2013). We present an approximation algorithm that achieves a guarantee of 2. Interestingly, we also show that for any constant $\epsilon > 0$, a $(2-\epsilon)$-ratio for MWEC implies that the Small Set Expansion Conjecture (Raghavendra and Steurer, 2010) does not hold. Thus, assuming the Small Set Expansion Conjecture, the bound of 2 is tight. In the FCEC problem, we are given a vertex weighted graph, a bound k, and our goal is to find a subset of vertices U of total weight at least k such that the number of edges with at least one edge in U is minimized. A $2(1 + \epsilon)$-approximation for the problem follows from the work of Carnes and Shmoys (2008). We improve the approximation ratio by giving a 2-approximation algorithm for the problem and show a $(2 - \epsilon)$-inapproximability under Small Set Expansion Conjecture. Only the NP-hardness result was known for this problem (Goldschmidt and Hochbaum, 1997). We show that a natural linear program for FCEC has an integrality gap of $2 - o(1)$. We also show that for any constant $\rho > 1$, an approximation guarantee of ρ for the FCEC problem implies a $\rho(1 + o(1))$ approximation for MWEC. Finally, we define the Degrees density augmentation problem which is the density version of the FCEC problem. In this problem we are given an undirected graph $G = (V, E)$ and a set $U \subseteq V$. The objective is to find a set W so that $(e(W) + e(U, W))/deg(W)$ is maximum. This problem admits an LP-based exact solution (Chakravarthy et al., 2012). We give a combinatorial algorithm for this problem.

1. Introduction

Given a graph $G = (V, E)$ and a subset $S \subseteq V$, let $deg(S)$ denote sum of degrees of all vertices in S and let $e(S, \bar{S})$ denote the number of edges that have one endpoint in S and the other in $V \setminus S$. Then the edge expansion, $\phi_G(S)$ is given by

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$\phi_G(S) = e(S, \overline{S})/deg(S)$. Given some δ, $0 < \delta \le 1/2$ and a d-regular graph F, let $\mathcal{S}$ denote all subsets of V of size $\delta|V|$. Let

$$\phi_G(\delta) = \min_{S \in \mathcal{S}} \frac{e(S, \overline{S})}{deg(S)}.$$

The Small Set Expansion Conjecture states that for any constant η, it is NP-hard to distinguish whether $\phi_G(\delta) \ge 1 - \eta$ or $\phi_G(\delta) \le \eta$. In [16], Raghavendra and Steurer showed that proving the Small Set Expansion Conjecture implies a proof for the Unique Games Conjecture and an algorithm that refutes the Unique Games Conjecture refutes the Small Set Expansion Conjecture.

In this paper we relate the Small Set Expansion Conjecture to two other edge expansion problems. We say that an edge e is *touched* by a set of vertices U or that e *touches* the set of vertices U, if at least one of e's endpoints is in U. Specifically, the problems that we study are as follows. The Maximum weight m'-edge cover (MWEC)problem that we study was first introduced by Goldschmidt and Hochbaum [10]. In this problem, we are given an undirected simple graph $G = (V, E)$ with integral vertex weights. The goal is to select a subset $U \subseteq V$ of maximum weight so that the number of edges touching U is at most m'. This problem is motivated by application in loading of semi-conductor components to be assembled into products [10].

We also study the closely related Fixed cost minimum edge cover (FCEC) problem in which given a graph $G = (V, E)$ with vertex weights and a number W, our goal is to find $U \subseteq V$ of weight at least W such that the number of edges touching U is minimized.

Finally, we study the Degrees density augmentation problem which is the density version of the FCEC problem. In the Degrees density augmentation problem, we are given an undirected graph $G = (V, E)$ and a set $U \subseteq V$ and our goal is to find a set W with maximum augmenting density i.e., a set W that maximizes $(e(W) + e(U, W))/deg(W)$.

1.1. Related work

Goldschmidt and Hochbaum [10] introduced the MWECproblem. They show that the problem is NP-complete and give algorithms that yield 2-approximate and 3-approximate algorithm for the unweighted and the weighted versions of the problem, respectively. Their NP-hardness proof applies to FCEC as well. Liang [14] improved the bound of 3 to $2 + \epsilon$, for any fixed $\epsilon > 0$.

A class of related problems are the density problems — problems in which we are to find a subgraph and the objective function considers the ratio of the total number or weight of edges in the subgraph to the number of vertices in the subgraph. A well known problem in this class is the Dense k-subgraphproblem (DkS) in which we want to find a subset of vertices U of size k such that the total number of edges in the subgraph induced by U is maximized. The best ratio known for the problem is $n^{1/4+\epsilon}$ [6,2], which is an improvement over the bound of $O(n^{1/3-\epsilon})$, for ϵ close to 1/60 [6]. The Dense k-subgraphproblem is APX-hard under the assumption that NP problems cannot be solved in subexponential time [11]. Interestingly, if there is no bound on the size of U then the problem can be solved in polynomial time [13,8].

The FCEC problem is related to the minimum knapsack problem [4,3]. In the minimum knapsack problem, we are given a set of items, and each item has a value and a cost. The objective is to select a minimum cost subset of items, S, such that the value of S is at least as large as the specified demand. If we associate with each vertex u, a cost of $deg(u)$, a value $w(u)$, and consider the objective function of minimizing $deg(U)$, the problem reduces to minimum knapsack problem, for which a $(1 + \epsilon)$-approximation follows from the work of Carnes and Shmoys [3]. Using this result and the observation that the objective function is at most a factor of 2 away from the objective function for the FCEC problem, a $2(1 + \epsilon)$-approximation follows for the FCEC problem.

Variations of the Dense k-subgraphproblem in which the size of U is at least k ($Dalk$) and the size of U is at most k ($Damk$) have been studied [1,12]. They give evidence that $Damk$ is just as hard as DkS. They also give 2-approximate solutions to the $Dalk$ problem. In [12], they also consider the density versions of the problems in directed graphs. Gajewar and Sarma [7] consider a generalization in which we are given a partition of vertices $U_1, U_2, \ldots, U_t$, and non-negative integers $r_1, r_2, \ldots, r_t$. The goal is to find a densest subgraph such that partition U_i contributes at least r_i vertices to the densest subgraph. They give a 3-approximation for the problem, which was improved to 2 by Chakravarthy et al. [5], who also consider other generalizations. They also show using linear programming that the Degrees density augmentationproblem can be solved optimally.

A problem parameterized by k is Fixed Parameter Tractable [15], if it admits an exact algorithm with running time of $f(k) \cdot n^{O(1)}$. The function f can be exponential in k or larger. Proving that a problem is W[1]-hard (with respect to parameter k) is a strong indication that it has no FPT algorithm with parameter k (similar to NP-hardness implying the likelihood of no polynomial time algorithm). The FCEC problem parameterized by k is W[1] hard but admits a $f(k, \epsilon) \cdot n^{O(1)}$ time, $(1 + \epsilon)$-approximation, for any constant $\epsilon > 0$ [15]. This is in contrast to our result that shows that it is highly unlikely that FCEC admits a polynomial time approximation scheme (PTAS), if the running time is bounded by a polynomial in k.

1.2. Preliminaries

The input is an undirected simple graph $G = (V, E)$ and vertex weights are given by $w(\cdot)$. Let $n = |V|$ and $m = |E|$. For any subset $S \subseteq V$, let $\overline{S} = V \setminus S$. Let $E(P, Q)$ be the set of edges with one endpoint in P and the other in Q. Let $deg(S)$ denote