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Weakly bipancyclic bipartite graphs

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ABSTRACT

We investigate the set of cycle lengths occurring in bipartite graphs with large minimum degree. A bipartite graph is *weakly bipancyclic* if it contains cycles of every even length between the length of a shortest and a longest cycle. In this paper, it is shown that if $G = (V_1, V_2, E)$ is a bipartite graph with minimum degree at least $n/3 + 4$, where $n = \max\{|V_1|, |V_2|\}$, then G is a weakly bipancyclic graph of girth 4. This improves a theorem of Tian and Zang (1989), which asserts that if G is a Hamilton bipartite graph on $2n (n \geq 60)$ vertices with minimum degree greater than $2n/5 + 2$, then G is bipancyclic (i.e., G contains cycles of every even length between 4 and $2n$). By combining the main result of our paper with a theorem of Jackson and Li (1994), we obtain that every 2-connected k-regular bipartite graph on at most $6k - 38$ vertices is bipancyclic.

1. Introduction

We investigate the set of cycle lengths occurring in bipartite graphs with large minimum degree. The *circumference* of a graph G is the length of a longest cycle, denoted by $c(G)$, and the *girth* is the length of a shortest cycle. A graph of order n is Hamiltonian if it has circumference n. A bipartite graph G is *weakly bipancyclic* if it contains cycles of every even length between the girth and the circumference. If, in addition, G has girth 4 and circumference $|G|$, it is said to be *bipancyclic*.

There are many results on the cycle structure of graphs. Among them are the following two theorems.

Theorem 1.1 (*Amar, Flandrin, Fournier and Germa [1]*). *Let G be a non-bipartite Hamiltonian graph of order $n \geq 102$. If $\delta(G) > 2n/5$, then G contains cycles of every length between 3 and n, and the bound is sharp.*

Theorem 1.2 (*Brandt [4]; Brandt, Faudree and Goddard [5]*). *Every non-bipartite graph of order n with minimum degree at least $(n + 2)/3$ contains cycles of every length between 4 and the circumference.*

In graph theory it is common for results to have a "bipartite" version; such a typical example is Jackson's theorem [9], which asserts that every 2-connected k-regular graph with at most $3k$ vertices is Hamiltonian. Häggkvist [7] conjectured that every 2-connected k-regular bipartite graph G with at most $6k$ vertices is Hamiltonian, which was confirmed by Jackson and Li [10] when G contains at most $6k - 38$ vertices. So a natural question to be asked is: what are the counterparts of the above two minimum degree theorems on bipartite graphs? A bipartite version of Theorem 1.1 has been established.

Theorem 1.3 (*Tian and Zang [13]*). *If G is a Hamiltonian bipartite graph on $2n$ vertices with minimum degree $\delta(G) > 2n/5 + 2$, where $n \geq 60$, then G is bipancyclic.*

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However, Mitchem and Schmeichel [11] made the following conjecture.

Conjecture 1.1 (*Mitchem and Schmeichel [11]*). *If G is a Hamiltonian bipartite graph on* $2n$ *vertices with minimum degree* $\delta(G) > (1 + \sqrt{4n-3})/2$, *then G is bipancyclic, and the bound is best possible.*

Unfortunately, Conjecture 1.1 was proved to be losing [12]. In this paper we establish the following bipartite version of Theorem 1.1, which improves Theorem 1.3.

Theorem 1.4. *Every Hamiltonian bipartite graph on* $2n$ *vertices with minimum degree at least* $n/3 + 4$ *is bipancyclic.*

Observe that if $G = (V_1, V_2, E)$ is a Hamiltonian bipartite graph on $2n$ vertices, then $c(G) = 2n$ and $|V_1| = |V_2| = n$. The theorem that we will actually prove, Theorem 1.5 on weakly bipancyclic graphs, is a little stronger than Theorem 1.4.

Theorem 1.5. *If* $G = (V_1, V_2, E)$ *is a bipartite graph with minimum degree at least* $n/3 + 4$, *where* $n = \max\ \{|V_1|, |V_2|\}$, *then G is a weakly bipancyclic graph of girth* 4.

It is interesting to note that there is no connectivity requirement in Theorem 1.5; nor is there any requirement on the difference between $|V_1|$ and $|V_2|$. By combining Theorem 1.4 with a theorem of Jackson and Li [10], we get the following result.

Corollary 1.6. *Every* 2*-connected k-regular bipartite graph on at most* $6k - 38$ *vertices is bipancyclic.*

Throughout this paper, we use C_k to denote a cycle of length k and set

$$\mathcal{H}_{n,\leq} = \{G : G = (V_1, V_2, E) \text{ is a 2-connected bipartite graph with } \delta(G) \geq n/3 + 4, \text{ where } n = \max\ \{|V_1|, |V_2|\}\}.$$

Our proof of Theorem 1.5 is organized as follows, the main part of which is to show that every graph G in $\mathcal{H}_{n,\leq}$ is weakly bipancyclic, where $n \geq 18$.

- In Section 2, we establish some basic lemmas on 2-connected bipartite graphs that will be used in Sections 4–6, one of which says that every two vertices in a 2-connected bipartite graph are connected by a path of length at least $2d - 2$, where d is the second minimum degree, that is, $d = \min\ \{d(v) :\ v \neq x,\ \text{where } x \text{ is a vertex with minimum degree}\}$.
- In Section 3, we define a set $\mathcal{C}^2_{2k}$ of graphs of order $2k$, where $k \geq 4$. Each graph of $\mathcal{C}^2_{2k}$, called *a tricycle of order* $2k$, contains three nested cycles C_{2k}, C_{2k-2}, and C_{2k-4} simultaneously. Our proof in this and the next section relies heavily on this nested cycle structure. We start with a tricycle of order 8 in $G \in \mathcal{H}_{n,\leq}$ and enlarge the tricycle we find by 2, 4, or 6 vertices. This process stops once we find a tricycle D of order at least $2(\delta(G) - 1)$, where $G - D$ consists of components with small diameter and large degree sum. It follows that G contains C_{2k} for $2 \leq k \leq \lceil n/3 \rceil + 3$.
- In Section 4, we use the so-called "weak connectivity", defined before Proposition 4.1, to show the existence of all even cycles C_{2k} with $\lceil n/3 \rceil + 4 \leq k \leq k^*$, where $k^* = \min\{2\lceil n/3 \rceil, |V_1|, |V_2|\}$. Let $G = (V_1, V_2, E) \in \mathcal{H}_{n,\leq}$, where $n \geq 18$. It is shown that G contains C_{2k} for $7 \leq k \leq 2\lceil n/3 \rceil$ if the weak connectivity of G is at most 6; otherwise, G contains a tricycle of order between $2k + 2$ and $2k + 6$ for each integer k with $\lceil n/3 \rceil \leq k < k^*$.
- In Section 5, we use the tool "segmentally insertible path", defined in Section 2.3, to show that if $G = (V_1, V_2, E)$ is a graph in $\mathcal{H}_{n,\leq}$ with $c(G) > 2k^*$, where $n \geq 18$, then G contains cycles of all even lengths between $2k^*$ and the circumference. For this purpose, we first design an algorithm to produce a set of segmentally insertible paths, and then employ it to show that if G is a graph in $\mathcal{H}_{n,\leq}$ containing a C_{2k}, then it contains either a C_{2k-2} or a C_{2k-4} with $|E(G - C_{2k-4})| \geq 2$, where $k \geq 2\lceil k/3 \rceil + 2$.
- In Section 6, we prove Theorem 1.5 and propose a conjecture on weakly bipancyclic bipartite graphs.

2. Preliminaries

In this section, we establish some basic lemmas on bipartite graphs that will be used in Sections 4–6. Let us introduce some notions before further discussion.

Given a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set respectively. For $v \in V(G)$, we use $d(v)$ and $N(v)$ to denote its degree and neighborhood respectively. For each subgraph H of G, we set $N_H(v) = N(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$. When G is a bipartite graph with bipartition (V_1, V_2), we set $V_i(H) := V_i \cap V(H)$ for $i = 1, 2$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S and set $G - S = G[V(G) - S]$.

For a path or cycle R, we use $\ell(R)$ to denote the length of R and assume that R has a given orientation. With this assumption, v^+_R and v^-_R will stand for the successor and predecessor of a vertex v on R under this orientation, respectively; we shall drop the subscript R if there is no danger of confusion. We define v^{+i} recursively by $v^{+0} = v$ and $v^{+(i+1)} = (v^{+i})^+$ for $i \geq 0$, and define v^{-i} analogously. For any two vertices u and v on R, let $u \overrightarrow{R}\, v$ or $R[u, v]$ denote the path from u to v on R in the given direction. The same path with the opposite direction will be denoted by $v \overleftarrow{R}\, u$. Set $R[u, v) := R[u, v] - \{v\}$, and $R(u, v] := R[u, v] - \{u\}$, etc. For each $X \subseteq V(R)$ and $i \geq 1$, define $X^{+i} := \{x^{+i} : x \in X\}$ and $X^{-i} := \{x^{-i} : x \in X\}$. If $X = N_R(v)$ for some vertex v, then we shall simply write $N^{+i}_R(v)$ and $N^{-i}_R(v)$ as opposed to the more cumbersome $(N_R(v))^{+i}$ and $(N_R(v))^{-i}$. We also define $X^{+0} := X =: X^{-0}$ for convenience.

A *block* of G is a maximal connected subgraph of G that contains no cut vertex. Let G be a graph and B a block of G. We say that a vertex v of B is an *internal vertex* of B if v is not a cut vertex of G, and that B is an *end block* of G if B contains at most one cut vertex of G.