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Collapsible graphs and Hamilton cycles of line graphs

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ABSTRACT

For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. The reduction of G is the graph obtained from G by contracting all maximally collapsible subgraphs until no collapsible subgraph is left. Let G be a graph on $n \geq 8$ vertices. In this paper, we prove that if $d(x)+d(y) \geq n-2-p(n)$ for each $xy \in E(G)$, then G is collapsible or G is one of 43 special graphs or the reduction of G is $K_{1,t}$ where $t \geq 2$ or G is a class of well-characterized graphs, where $p(n) = 0$ for n even and $p(n) = 1$ for n odd, which generalizes the earlier results by Catlin (1987), and by Li and Yang (2012).

1. Introduction

The graphs we consider here are finite, connected simple graphs without loops. We follow [1] for terminology and notations unless stated otherwise.

For a vertex v of a graph G, let $N_G(v)$ denote the set of the neighbors of v in G and for $A \subseteq V(G)$, let $N_G(A)$ denote the set $\cup_{v\in A} N_G(v)\backslash A$. For a subgraph H of a graph G and $v \in V(G)$, denote by $d_H(v) = |N(v) \cap V(H)|$ the number of the neighbors of v in H. If $H = G$, then $d_G(v)$ is the degree of v and we write $d(v)$ for it. For two vertex-disjoint subgraphs A and B of G, let $e_G(A, B)$ (or briefly $e(A, B)$) denote the number of edges with one end in A and the other end in B.

The concept of collapsible graphs was introduced by Catlin in [4]. For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H_R such that $O(H_R) = R$. A graph is *eulerian* if it has a closed trail through all the edges of G, that is, G is eulerian if it is connected with $O(G) = \emptyset$. A graph G is *supereulerian* if G has a spanning eulerian subgraph. Note that every collapsible graph is supereulerian.

For a subgraph H of a graph G, the graph G/H is obtained from G by identifying the two ends of each edge in H and then deleting the resulting loops. The contraction G/H is called the *reduction* of G if H is the maximally collapsible subgraph of G, that is, there is no non-trivial collapsible subgraph in G/H. A vertex u in G/H is called *non-trivial* if the vertex is obtained by contracting a non-trivial collapsible subgraph H_u. We call H_u the *preimage* of u. A graph is *reduced* if it is the reduction of itself.

For a graph G and an edge $e = uv$, $d(u) + d(v)$ is called the *edge degree* of e. Define $\xi(G) = \min\{d(x) + d(y) : xy \in E(G)\}$. The function $p(n)$ is defined as follows: $p(n) = 0$ if n even and $p(n) = 1$ otherwise.

Thomassen [14] conjectured that every 4-connected line graph is Hamiltonian. Harary and Nash-Williams [11] proved the classic result that if a graph G is not a star, the line graph $L(G)$ is Hamiltonian if and only if G has a dominating closed trail. Clearly, both collapsible graphs and supereulerian graphs have dominating closed trails. It is well-known that the line graph of 4-edge-connected graph is Hamiltonian. Thus, the collapsibility and supereulerianicity of k-edge-connected graphs where $k \in \{1, 2, 3\}$ are investigated (see [5,7–9,12,15,16] and others). In this paper, we still ficus on the collapsibility of connected graphs.

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Fig. 1. G'_7 and G''_7.

Theorem 1.1 (*Catlin [3]*). *Let G be a connected graph on n vertices and let $u, v \in V(G)$. If $\xi(G) \geq n$, then exactly one of the following holds:*
(1) *G has a spanning (u, v)-trail.*
(2) *$d(z) = 1$ for some vertex $z \notin \{u, v\}$.*
(3) *$G = K_{2,n-2}$, $u = v$ and n is odd.*
(4) *$G = K_{2,n-2}$, $u \neq v$, $uv \notin E(G)$, where n is even and $d(u) = d(v) = n - 2$.*
(5) *$u = v$, and u is the only vertex with degree 1 in G.*

Theorem 1.1 was improved as follows.

Theorem 1.2. *Let G be a connected graph on $n \geq 4$ vertices. If $\xi(G) \geq n$, then exactly one of the following holds:*
(1) *G is collapsible.*
(2) *The reduction of G is $K_{1,t-1}$ for $t \geq 3$ such that all of the vertices of degree 1 are trivial and they have the same neighbor in G, $t \leq \frac{n}{2}$. Moreover, if $t = 2$, then $G - v$ is collapsible for a vertex v in the K_2.*
(3) *G is $K_{2,n-2}$.*

Theorem 1.2 was strengthened by Li and Yang [13] as follows.

Theorem 1.3. *Let G be a connected graph on $n \geq 4$ vertices. If $\xi(G) \geq n - 1 - p(n)$, then exactly one of the following holds:*
(1) *G is collapsible.*
(2) *The reduction of G is $K_{1,t-1}$ for $t \geq 3$ such that all of the vertices of degree 1 are trivial and they have the same neighbor in G, $t \leq \frac{n}{2}$. Moreover, if $t = 2$, then $G - v$ is collapsible for a vertex v in the K_2.*
(3) *G is one of $\{C_5, G'_7, G''_7, K_{1,n-1}, K_{2,n-2}, K'_{2,n-3}\}$, where $K'_{2,n-3}$ is obtained from $K_{2,n-3}$ by adding a pendant edge on one of the vertices of degree $n - 3$, where G'_7 and G''_7 are shown in Fig. 1.*

An edge is *pendant* if it is incident with a vertex of degree 1. Denote by $K''_{2,n-4}$ the graph obtained from $K_{2,n-4}$ by adding two pendant edges on one of the vertices of degree $n - 4$. Denote by $K'''_{2,n-5}$ the graph obtained from $K_{2,n-5}$ by adding three pendant edges on one of the vertices of degree $n-5$. Denote by $K^*_{2,n-4}$ the graph obtained from $K_{2,n-4}$ by adding one pendant edge on each vertex of degree $n - 4$. Denote by $K^*_{2,n-5}$ the graph obtained from $K_{2,n-5}$ by adding two pendant edges on one vertex of degree $n-5$ and adding one pendant edge on the other vertex of degree $n-5$. Denote by $K^*_{2,n-6}$ the graph obtained from $K_{2,n-6}$ by adding two pendant edges on each vertex of degree $n - 6$. Define a family $\mathcal{H}$ of graphs such that if $G \in \mathcal{H}$, then G is the graph obtained from two vertex disjoint graphs H_1 and H_2 by adding an edge with one endpoint in H_1 and the other in H_2. Moreover, let $|V(G)| = n$. If $n = 2l$, then $H_1 = H_2 = K_l$; if $n = 2l + 1$, then $H_1 = K_l$ and H_2 is the graph obtained from K_{l+1} by deleting a set of independent edges.

Motivated by Theorems 1.2 and 1.3, we present the following result in this paper.

Theorem 1.4. *Let G be a connected graph on $n \geq 8$ vertices. If $\xi(G) \geq n - 2 - p(n)$, then exactly one of the following holds:*
(1) *G is collapsible.*
(2) *The reduction of G is $K_{1,t-1}$ for $t \geq 3$ such that all of the vertices of degree 1 are trivial and they have the same neighbor in G, $t \leq \frac{n}{2}$. Moreover, if $t = 2$, then $G - v$ is collapsible for a vertex v in the K_2.*
(3) *$G = G_i$, where $1 \leq i \leq 35$, or G is one of $\{K_{1,n-1}, K_{2,n-2}, K'_{2,n-3}, K''_{2,n-4}, K^*_{2,n-4}, K'''_{2,n-5}, K^*_{2,n-5}, K^*_{2,n-6}\}$.*
(4) *G is a class of well-characterized graphs $\mathcal{H}$.*

The paper is organized as follows: In Section 2, the former related results are presented and some lemmas are established. In Section 3, Theorem 1.4 is proved. Applications of Theorem 1.4 will be presented in Section 4.

2. Lemmas

Reduction method will play a key role in our proof of Theorem 1.4. It is known that any cycle of length less than 4 is collapsible. Some of the previous results concerning collapsible graphs in [4] are summarized as follows:

Theorem 2.1. *Let G be a connected graph. Each of the following holds:*
(1) *If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible; G is supereulerian if and only if G/H is supereulerian.*
(2) *A reduced graph does not have a cycle of length less than 4.*
(3) *If G is reduced and $|E(G)| \geq 3$, then $\delta(G) \leq 3$ and $2|V(G)| - |E(G)| \geq 4$.*
(4) *If G is reduced and violates (3), then G is either K_1 or K_2.*