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Distance k-domination, distance k-guarding, and distance k-vertex cover of maximal outerplanar graphs[☆]

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ABSTRACT

For positive integers k and n, let $\gamma_k(n)$, $g_k(n)$, and $\beta_k(n)$ denote the maximum values of the distance k-domination number, the distance k-guarding number, and the distance k-vertex cover number of maximal outerplanar graphs of order n, respectively. Known results imply $\gamma_1(n) = \lfloor n/3 \rfloor$ for $n \geq 3$ (Matheson et al., 1996), $g_1(n) = \lfloor n/3 \rfloor$ for $n \geq 3$ (Chvátal, 1975), $\gamma_2(n) = g_2(n) = \lfloor n/5 \rfloor$ for $n \geq 5$, and $\beta_2(n) = \lfloor n/4 \rfloor$ for $n \geq 4$ (Canales et al., 2015).

We show $\gamma_k(n) = g_k(n) = \lfloor n/(2k+1) \rfloor$ for $n \geq 2k+1$, and $\beta_k(n) = \lfloor n/2k \rfloor$ for $n \geq 2k \geq 4$.

1. Introduction

We consider finite, simple, and undirected graphs and use standard terminology.

Let G be a graph, k be a positive integer, and D be a set of vertices of G. The set D is a *distance k-dominating set* of G if for every vertex u of G, there is a vertex v in D with $\text{dist}_G(u, v) \leq k$. The minimum cardinality of a distance k-dominating set of G is the *distance k-domination number* $\gamma_k(G)$ of G. The set D is a *distance k-vertex cover* of G if for each edge e of G, there is a path of length at most k that contains e and a vertex from D. The minimum cardinality of a distance k-vertex cover is the *distance k-vertex cover number* $\beta_k(G)$ of G. Note that classical vertex covers coincide with distance 1-vertex covers. Now let G be planar with a fixed embedding. The set D is a *distance k-guarding set* of G if for every face f of G, there is a vertex u in the boundary of f, and a vertex v in D with $\text{dist}_G(u, v) \leq k - 1$. The minimum cardinality of a distance k-guarding set of G is the *distance k-guarding number* $g_k(G)$ of G.

While the distance k-domination number is a known variation of the classical domination number [5], the distance k-vertex cover number as well as the distance k-guarding number were introduced by Canales et al. [2]. The notion of graph guarding originates from a problem posed by Victor Klee and solved by Chvátal [3], which concerns the minimum number of guards needed to control an art gallery. Since we only consider distance variants of domination, vertex cover, and guarding, we omit the word 'distance' in their names for simplicity.

In the present paper, we study the maximum values of the k-domination number, the k-vertex cover number, and the k-guarding number of maximal outerplanar graphs of a given order. Therefore, for positive integers k and n, let

$$\gamma_k(n) := \max\{\gamma_k(G) : G \text{ is maximal outerplanar and has order } n\},$$

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$$\beta_k(n) := \max\{\beta_k(G) : G \text{ is maximal outerplanar and has order } n\} \quad \text{and}$$
$$g_k(n) := \max\{g_k(G) : G \text{ is maximal outerplanar and has order } n\}.$$

Several values follow from known results. Chvátal's theorem [3] implies $g_1(n) = \left\lfloor \frac{n}{3} \right\rfloor$ for $n \geq 3$. A short proof of Chvátal's theorem was given by Fisk [4]. Matheson and Tarjan [6] studied the domination number of planar graphs whose bounded faces are all bounded by triangles. Their result implies $\gamma_1(n) = \lfloor n/3 \rfloor$ for $n \geq 3$. A more refined bound on the domination number of maximal outerplanar graphs was given by Campos and Wakabayashi [1]. Recently, Canales et al. [2] showed $\gamma_2(n), g_2(n) = \lfloor n/5 \rfloor$ for $n \geq 5$ and $\beta_2(n) = \lfloor n/4 \rfloor$ for $n \geq 4$. Variating extremal outerplanar graphs described in [6], Canales et al. [2] construct graphs that show $\gamma_k(n), g_k(n) \geq \left\lfloor \frac{n}{2k+1} \right\rfloor$ whenever n is an even multiple of $2k+1$, and $\beta_k(n) \geq \left\lfloor \frac{n}{2k} \right\rfloor$ whenever n is an even multiple of $2k$.

Our main results are the following.

Theorem 1. *If k and n are positive integers with $n \geq 2k+1$, then $\gamma_k(n) = g_k(n) = \left\lfloor \frac{n}{2k+1} \right\rfloor$.*

Theorem 2. *If k and n are positive integers with $n \geq 2k$ and $k \geq 2$, then $\beta_k(n) = \left\lfloor \frac{n}{2k} \right\rfloor$.*

The rest of the paper is devoted to the proofs of these theorems.

2. Proofs of Theorems 1 and 2

For every planar graph that we consider, we will tacitly assume that it has a fixed embedding in the plane. Furthermore, for every outerplanar graph, we assume that it is embedded in such a way that all its vertices are on the boundary of the unbounded face. This implies that the boundary of every bounded face of a maximal outerplanar graph is a triangle.

Let G be a maximal outerplanar graph of order at least 3. The boundary of the unbounded face of G is a Hamiltonian cycle C of G. A *chord* of G is an edge of G that does not belong to C. Let uv be a chord of G. The *length* $\ell_G(uv)$ of uv is the distance $\text{dist}_C(u, v)$ between u and v within C. Adding uv to the cycle C results in a graph that has two cycles C_1 and C_2 that are distinct from C. Furthermore, C_1 and C_2 are the boundaries of two maximal outerplanar subgraphs of G whose union is G and whose intersection is the edge uv. We will refer to these two graphs as the *subgraphs of G generated by uv*. We refer to the edges of some graph G as *G-edges*.

The following lemma was observed in [2]. For the sake of completeness, we include the short proof.

Lemma 3 (*Canales et al. [2]*)**.** *If G is a maximal outerplanar graph of order at least 3 and k is a positive integer, then $\gamma_k(G) \leq g_k(G) \leq \beta_k(G)$.*

Proof. Let D be a k-guarding set of G and let u be a vertex of G. Since G is maximal outerplanar and has order at least 3, the vertex u belongs to a face f of G whose boundary is a triangle. Since D is a k-guarding set of G, there is a vertex u' in the boundary of f, and a vertex v in D with $\text{dist}_G(u', v) \leq k-1$. Since u is either equal to u' or a neighbor of u', we have $\text{dist}_G(u, v) \leq k$, which implies that D is a k-dominating set of G. This implies $\gamma_k(G) \leq g_k(G)$. A similar argument shows that every k-vertex cover of G is also a k-guarding set of G, which implies $g_k(G) \leq \beta_k(G)$. □

Clearly, Lemma 3 implies that $\gamma_k(n) \leq g_k(n) \leq \beta_k(n)$.

Lemma 4. *Let k and n be positive integers.*

(i) $\gamma_k(n+1) \geq \gamma_k(n)$.
(ii) $\beta_k(n+1) \geq \beta_k(n)$.
(iii) $g_k(n+1) \geq g_k(n)$.

Proof. We give details only for (iii). For (i) and (ii), similar arguments work. Clearly, we may assume that $n \geq 3$. Let G be a maximal outerplanar graph of order n with $g_k(G) = g_k(n)$. Let C be the boundary of the unbounded face of G. Let uv be an edge of C, and let G' arise from G by adding a new vertex x embedded in the unbounded face of G and adding the two new edges xu and xv to G. Clearly, G' is a maximal outerplanar graph of order $n+1$. The boundary of the unbounded face of G' arises from C by replacing uv with uxv. Let D' be a k-guarding set of G' of order $g_k(G')$. If $x \notin D'$, then D' is a k-guarding set of G. If $x \in D'$, then $(D' \setminus \{x\}) \cup \{u\}$ is a k-guarding set of G. This implies $g_k(n+1) \geq g_k(G') \geq g_k(G) = g_k(n)$. □

The following lemma was proved by Shermer [7]. For the sake of completeness, we include the short proof.

Lemma 5 (*Shermer [7]*)**.** *Let s and n be positive integers with $s \geq 2$. Let G be a maximal outerplanar graph of order n and let C be the boundary of the unbounded face of G. If $n \geq 2s$, then G has a chord uv such that one of the subgraphs of G generated by uv has m C-edges where $s \leq m \leq 2s-2$.*

Proof. Since G has a vertex of degree 2, there is a chord e of G of length 2. Clearly, one of the subgraphs of G generated by e has $n-2$ C-edges. Note that $n-2 \geq s$. Let the chord uv of G be chosen such that one of the subgraphs of G generated by